BEM ANALYSIS OF CONTACT PROBLEMS

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CERTIFICATE

This is to certify that the thesis entitled, 'BEM Analysis of Contact Problems' by K.G. SREERENGARAJAN, is a bonafide recor of work done by him under our guidance and supervision for the award of the degree of Master of Technology in the Indian Institute of Technology, Kanpur. The work carried out in this thesis has not been submitted elsewhere for the award of a degree.

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CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

A	Body A
a	Sampling point
B	Body B
c	Contact region
ca	No-slip region of the contact zone
cs	Slip region of the contact zone
g,h	Submatrices of size 2x2
E	Young's modulus
Fig.1.1	Figure number
G	Shear modulus
G ,H	Assembled matrices
i	Source point
j	j^{th} element, directions 1 and 2
k,l	Directions 1 and 2
m	Number of a node
N	Total number of the nodes
n	Normal to boundary
n_k	Normal in k direction
P_k	Applied load in k direction/elemental length
p_k	Traction in k direction
q	Element
r	Distance between source point and the sampling point
r_k	r in k direction
Table 4.1	Table number
t_k	Traction in k direction

u_k	Displacement in k direction
v_k	Displacement in k direction of the deformed body in the contact region
x, y, z	Cartesian co-ordinates
δ_1^i	Unit load in 1 direction at point i (Dirac delta)
δ_x	Relative displacement in no-slip region
η_1, η_2	Natural co-ordinates
ϕ	Shape functions
$\bar{\Gamma}$	Boundary
μ	Sliding coefficient of friction
μ_R	Rolling coefficient of friction
ν	Poisson's ratio
Ω	Domain
σ_{jk}	Stress tensor
Σ	summation symbol
$\mid t_x \mid$	Mod value of t_x
$\bar{u}$ $\bar{p}$	Known values of u and p
*	Weighting field
[1]	Reference 1

ABSTRACT

When two or more bodies are in contact the load transfer takes place through the contact areas which are very small usually and the stresses in the contact region are very high. Plastic flow as well as fatigue failures may occur due to these high amount of stresses. Boundary Element Method is particularly suitable for contact problems and has been used to analyse such problems, whose boundary conditions are non-linear. Both static and rolling contact problems are analysed. Linear elastic bodies in two dimension are analysed with plane stress or plane strain assumption. Constant elements and linear elements are used for the numerical solution. Friction in the contact region is taken into consideration. Boundary conditions for static and rolling contact problems are discussed. A method to determine the contribution of micro-slip in the contact region towards the overall rolling friction coefficient is discussed.

CHAPTER 1

INTRODUCTION

Contact problems and the study of the load transfer in mechanical assemblages like rail and railwheel, cam mechanism, journal bearings, etc. are of great importance due to the severity of the stresses in the vicinity of contact region. If the stress exceeds the linear range there will be plastic flow. Moreover if cyclic stresses are applied then to find out the fatigue strength of the material, an analysis of the contact region becomes imperative.

There are two types of contact problems: (i) Static contact problems and (ii) Rolling contact problems. Several analytical solutions for half planes are available. Hertz solutions [1] are for static conditions without friction, whereas solutions by Smith and Liu [2] consider friction also in the contact region. Poritsky [3] considers friction and discusses the boundary conditions in the contact region for rolling contact. A review by Johnson [4] gives a comprehensive account of the nature of contact problems.

Later many other solutions had been obtained using numerical methods like Finite Element Methods (FEM) [5,6,7] and Boundary Element Methods [8]. In FEM most of the available solutions are for half-plane problems [5,7]. In all these

cases traction distributions along the contact region and transverse to the contact region are found out by an iterative procedure in which starting traction along the contact region is chosen to be zero. In the subsequent steps it is increased slightly and the resulting traction distribution along the contact region is supplied as the new distribution. Iterations are continued till tractions converge. Linear programming techniques were also used by treating the contact conditions as constraints [6] . A solution of a contact problem is basically a solution of two or more Navier's equations with contact conditions. With appropriate boundary conditions both static and rolling contact types of problems can be solved.

Boundary Element Method (BEM), a method which finds increasing applications in special problems like infinite domain, fracture mechanics, etc., can also be used for contact problems [8,9] . The method typically starts with weighted residual statement of the equilibrium equation. The weighting functions are Green's solutions of the equilibrium equation. These weighting functions are of such nature that the domain terms in the weighted residual statement disappears and only the boundary terms remain as an integral equation. Similar integral equations are obtained for each of the bodies involved in the contact. These equations can be solved using the contact conditions to obtain the unknown values on the boundary. Using these boundary values internal stresses, if necessary, can be calculated.

BEM has the following distinct advantages over the domain methods such as FEM:

(i) BEM treats only the boundaries which are of primary interest in the solution procedure.

(ii) It is possible to couple normal and tangential tractions.

(iii) The contact pressures are obtained directly from the tractions which are primary unknown quantities and are determined with the same accuracy as the displacement unknowns.

(iv) Sub-surface stresses are calculated, if they are needed, and they are also more accurate than displacement FEM formulation.

In this work an attempt is made to determine the rolling friction while circular wheel rolls on a horizontal ground. The rolling friction is due to several mechanisms of energy dissipation such as micro-slip in the contact region, hysteresis losses, impact, etc. [10]. The present work attempts to determine the rolling friction due to micro-slip [11,12] in the contact region. Contact problems in 2-D with linear materials are analysed. Mainly cylinder on an elastic foundation is considered, with different combination of materials under different loading conditions.

In Chapter 2, BEM formulation for isotropic, two dimensional problems has been described. Chapter 3 describes the BEM formulation for contact problems and presents the various sets of contact conditions. The solution procedure of

the contact problems is also discussed. Chapter 4 discusses the results while Chapter 5 gives conclusions.

CHAPTER 2

BEM FORMULATION FOR ISOTROPIC BODIES

The procedure of applying Boundary Element Method to linearly elastic problems is given in the following sections. Two dimensional isotropic bodies are considered unless otherwise stated. Numerical implementation and the method of stress evaluation are also presented.

2.1 Fundamental Solutions:

Equilibrium equations for two dimensional isotropic bodies in the absence of body forces can be written as

$$\sigma_{jk,j} = 0 \tag{2.1}$$

where σ_{jk} is the stress tensor.

If Γ is the boundary of the domain then the boundary conditions, displacements u_j and tractions p_j, are

$$u_j = \bar{u}_j \text{ on } \Gamma_1 \quad j = 1,2$$

$$p_j = \bar{p}_j \text{ on } \Gamma_2 \quad j = 1,2 \tag{2.2}$$

and $\Gamma = \Gamma_1 + \Gamma_2$.

Fundamental solution for Eq. 2.1 is the solution of the following equation:

$$\sigma^{*}_{jk,j} + \delta^{i}_{1} = 0 \tag{2.3}$$

where δ^i_l represents a unit load at point 'i' in the direction l (Dirac delta function) and σ^*_{jk} is the stress tensor (Fig. 2.1). So, the fundamental solutions for the two dimensional isotropic bodies (plane strain) are [13,14]

$$u^*_{lk} = \frac{1}{8\pi G(1-\nu)}\left[(3-4\nu)\ln\frac{1}{r}\ \delta_{lk} + r,_l\, r,_k\right] \tag{2.4}$$

$$p^*_{lk} = \frac{-1}{4\pi(1-\nu)r}\left[\frac{\partial r}{\partial n}\left((1-2\nu)\,\delta_{lk} + 2\, r,_k\, r,_l\right) - (1-2\nu)\left(r,_l\, n_k - r,_k\, n_l\right)\right] \tag{2.5}$$

where

p^*_{lk} - tractions in k directions due to a unit force in the l direction

u^*_{lk} - displacements in k direction due to a unit force in the l direction

G - shear modulus

ν - Poisson's ratio

r - distance between the point where unit load is applied (source point, i) and an arbitrary point in the field (sampling point, a) (Fig. 2.1)

r_k - component of r in k direction

n - unit normal on the boundary (Fig. 2.1)

n_k - normal in k direction on the boundary

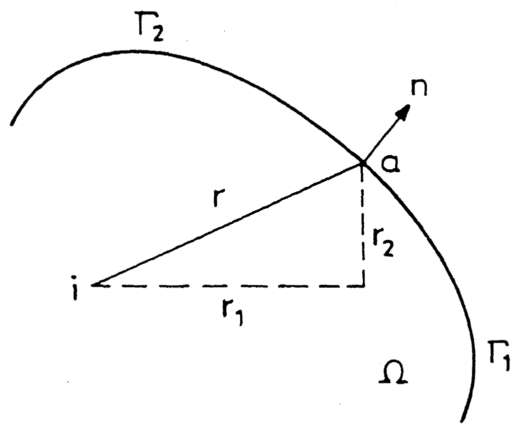


Fig. 2.1 Unit load at point i in an isotropic body.

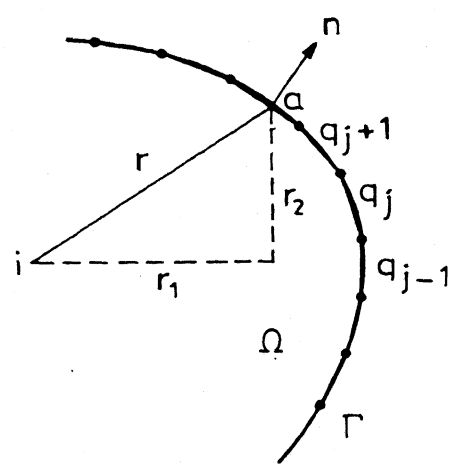


Fig. 2.2 BEM discretization.

Physically these fundamental solutions are the resulting displacements and tractions in k direction due to a unit load applied at a point i in an infinite body in l direction [15].

2.2 Weighted Residual Statement:

Considering the unit force is applied in one particular direction, the principle of virtual displacements for linear elastic problems can be written as [13]

$$\int_{\Omega} \sigma_{jk,j} \; u_k^* \; d\Omega = \int_{\Gamma_2} (p_k - \bar{p}_k) \, u_k^* \, d\Gamma \tag{2.6}$$

where u_k^* are the virtual displacements identically satisfying the homogeneous boundary conditions $u_k^* \equiv 0$ on Γ_1.

If we now interpret u_k^* as weighting functions which do not satisfy these conditions on Γ_1 then the expression can be rewritten as,

$$\int_{\Omega} \sigma_{jk,j} \; u_k^* \; d\Omega = \int_{\Gamma_2} (p_k - \bar{p}_k) u_k^* \, d\Gamma + \int_{\Gamma_1} (\bar{u}_k - u_k) p_k^* \, d\Gamma \tag{2.7}$$

where

$$p_k^* = \sigma_{jk}^* \, n_j \; .$$

2.3 Boundary Element Equation:

Integrating Eqn. 2.7 by parts twice and substituting Eqn. 2.3, we get

$$u_1^i + \int_\Gamma u_k \; p_k^* \; d\Gamma = \int_\Gamma p_k \, u_k^* \; d\Gamma \qquad (2.8)$$

where u_j^i represents the displacement at point i in the 1 direction.

If we consider unit forces are acting in the two directions (three directions for 3-dimensional problems) the above equation becomes

$$u_1^i + \int_\Gamma u_k \, p_{1k}^* \; d\Gamma = \int_\Gamma p_k \, u_{1k}^* \; d\Gamma \qquad (2.9)$$

This equation is the basic relation for developing the BEM models. In this equation the first term provides displacement of a point in the domain whereas all other terms relate to the points on the boundary. Eqn. 2.9 can be modified so that the first term refers to the displacement of a point on the boundary by assuming a small hemisphere around the point and letting the size of the sphere go to zero [3]. Applying the limiting procedure Eqn. 2.9 becomes

$$c^i \; u_1^i + \int_\Gamma u_k \, p_{1k}^* \; d\Gamma = \int_\Gamma p_k \, u_{1k}^* \; d\Gamma \qquad (2.10)$$

where the value of c^i depends on the smoothness of the boundary. c^i contributes to Eqn. 2.10 only if the source point 'i' and sampling point 'a', Fig. 2.1, coincides.

2.4 Numerical Implementation:

The boundary is discretised into certain number of elements which are one-dimensional lines in the case of 2-D regions (surfaces in the case of 3-D regions). The elements

may be of the following types (Fig. 2.3):

(a) constant elements

(b) linear elements

(c) quadratic and higher order elements.

For constant elements the variables (u, p) will remain constant within each element and they will be referred with respect to the node which is at the centre of the element. For linear elements variables vary linearly and so on. Eqn. 2.10 in discretised form is

$$c^i u_l^i + \sum_{j=1}^{N} \int_{\Gamma_j} p_{lk}^{*} u_k \, d\Gamma = \sum_{j=1}^{N} \int_{\Gamma_j} u_{lk}^{*} p_k \, d\Gamma \qquad (2.11)$$

$$l = 1,2$$
$$k = 1,2$$

where N is the number of elements.

Γ_j element 'j' on the discretised boundary (Fig.2.2). The summation in the second and third terms refer to summation over all the elements on the boundary. Eqn. 2.11 is a set of 2-equations per node, corresponding to each of the degrees of freedom. In Eqn. 2.11 each term is a 2x2 matrix and

$$\text{First term} = \begin{bmatrix} c_{11}^i & c_{12}^i \\ c_{21}^i & c_{22}^i \end{bmatrix} \begin{Bmatrix} u_1^i \\ u_2^i \end{Bmatrix} = [c^i]\{u^i\}$$

where $[c^i]$ can have values only if i=j and the values vary according to the smoothness of the boundary, for example, on a smooth surface $c_{11}^i = c_{22}^i = 0.5$ and $c_{12}^i = c_{21}^i = 0.$

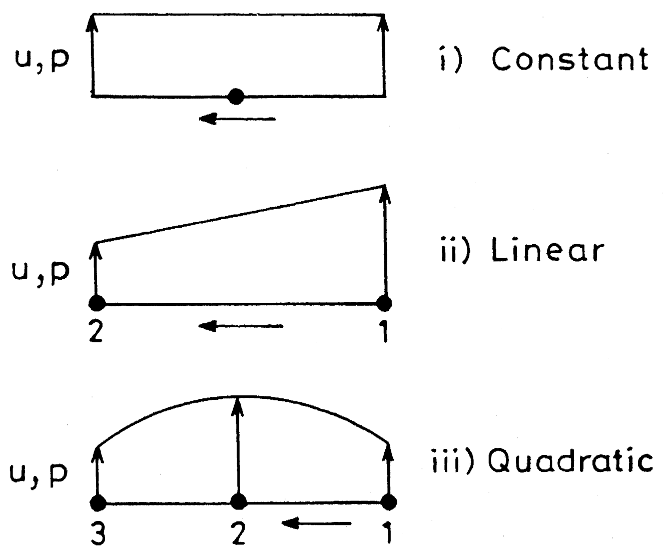


Fig. 2.3 Different types of elements.

Second term (of Eqn. 2.11) for one particular j

$$= \int_{\Gamma_j} \begin{bmatrix} p_{11}^* & p_{12}^* \\ p_{21}^* & p_{22}^* \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \, d\Gamma$$

Third term of Eqn. 2.11 for one particular j

$$= \int_{\Gamma_j} \begin{bmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \, d\Gamma$$

For constant elements u_k and p_k ($k = 1,2$) will remain constant in an element. In case of linear elements u_k and p_k will be varying within the element. They can be expressed using the nodal values and the shape functions as

$$\begin{aligned} u_k &= \phi^T u_k' \\ p_k &= \phi^T p_k' \end{aligned} \qquad k = 1,2 \tag{2.12}$$

where,

u_k' - nodal displacements of the element

p_k' - nodal tractions of the element

$\phi^T = \begin{bmatrix} \phi_1 & \phi_2 & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \end{bmatrix}$ (ϕ_1 and ϕ_2 are the shape functions for linear element)

Referring to Fig. 2.1, q_{j-1} and q_j are the nodes of j-1th element and q_j and q_{j+1} are the nodes of the jth element. It can be observed from Eqn. 2.11 and Eq. 2.12 that the variables connected with q_j will figure in the integration of both j-1th and jth elements. Similarly all the nodes will contribute to the adjacent elements. So, for each ith node when the integration is carried over the boundary a set of two equations with 2N variables under each integral term are obtained. For each node Eq. 2.11 gives a 2x2 matrix associated with the variables of node under each integral term. In matrix form Eq. 2.11 for source point i is

$$\begin{bmatrix} h_{11} & h_{12} \cdots h_{1\ 2i-1} & h_{1\ 2i} & \cdots & h_{1n} \\ h_{12} & h_{22} \cdots h_{2\ 2i-1} & h_{2\ 2i} & \cdots & h_{2n} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1 \\ \vdots \\ \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_i \\ \vdots \\ \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_N \end{Bmatrix}$$

$$= \begin{bmatrix} g_{11} & g_{12} \cdots g_{1\ 2i-1} & g_{1\ 2i} & \cdots & g_{1n} \\ g_{21} & g_{22} \cdots g_{2\ 2i-1} & g_{2\ 2i} & \cdots & g_{2n} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}_1 \\ \vdots \\ \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}_i \\ \vdots \\ \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}_N \end{Bmatrix} \qquad (2.13)$$

where h_{ij} is LHS of Eqn. 2.11

g_{ij} is RHS of Eqn. 2.11

$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_i$ and $\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}_i$ are displacements and tractions of node i.

It can be noted that c^i term has been added with the value given by the integral term in LHS when $i = j$ (Eqn. 2.11). The evaluation of c^i will be explained later. The submatrices associated with $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1$ and $\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}_1$, both are the variables of node 1, are $[h] = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$ and $[g] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$.
Similarly all the nodes have such sub-matrices (Eqn. 2.13). When Eqn. 2.11 is evaluated for all the boundary nodes in turn, matrix equation shown below is obtained:

$$[H]\{u\} = [G]\{p\} \tag{2.14}$$

where,

$[H]$ - a matrix of size 2Nx2N with N-submatrices $[h]$ of size 2x2

$[G]$ - a matrix of size 2Nx2N with N-submatrices $[g]$ of size 2x2

$\{u\}$ and $\{p\}$ - displacement and traction vectors as given in Eqn. 2.13.

2.5 Evaluation of [H] and [G] and Solution Method:

Inspection of fundamental solutions, Eqn. 2.4 and Eqn. 2.5, shows the singularity arises in Eq. 2.11 when r becomes zero, i.e., when source point 'i' itself is also the

sampling point. These singularities ($i = j$) contribute to the diagonal submatrices of size 2x2, Eqn. 2.13, to both [H] and [G] in Eqn. 2.14. When $i=j$ the first term of Eqn.2.11 also contributes to the diagonal submatrices of [H] . Being a logarithmic singularity right hand side integral (when $i = j$) will be analytically evaluated in the principal value sense without much difficulty. In case of diagonal submatrices of [H] no analytical method is attempted, but they are evaluated from rigid body motion principles. By giving unit displacements in all directions Eqn. 2.14 becomes

$$[H]\{U\} = 0 \tag{2.15}$$

where $\{U\}$ is a vector defining unit rigid displacement in all the directions. So, the diagonal submatrices of [H] will be,

$$[h_{ij}]_m = - \sum_{\substack{k=1,N \\ k\neq m}} [h_{ij}]_k \qquad \begin{matrix} i=1,2 \\ j=1,2 \end{matrix} \tag{2.16}$$

where

N - the number of nodes

m - the node which gives rise to the singularity.

All other non-diagonal elements of both the matrices are numerically evaluated using 'Gaussian Quadrature'. Hence all the coefficients of [H] and [G] will be known.

It may be noted that in Eqn. 2.11 out of the 4 variables u_k and p_k ($k = 1,2$) at each node two will be known boundary conditions. If there are N nodes after discretization there will be 2N unknowns associated with them. Bringing all the known values to right hand side and the unknown values to the left side Eqn. 2.14 becomes

$$[A]\{X\} = \{B\} \qquad (2.17)$$

where

$[A]$ - matrix of size 2Nx2N

$\{X\}$ - unknown quantities of u_k & p_k in the form of vector of size 2N

$\{B\}$ - column vector of size 2N, calculated from the known boundary values.

Solving Eqn. 2.17 with any standard method the unknowns of the u_k and p_k's on boundary are obtained and thus all the boundary values are known.

2.6 Determination of Stresses in the Domain:

After solving the system given by Eqn. 2.17 we will have all the boundary values, i.e., displacements and tractions. With the help of Eqn. 2.8 we can get displacements at any point in the domain. Using these internal points, we can find out strains and hence the stresses.

Alternate way to determine stresses directly is to differentiate Eqn. 2.8 at the internal point. Substituting the fundamental solutions in the resulting expression we can get an expression which will directly give us the stresses as [13,14]

$$\sigma_{ij} = \int_{\Gamma} D_{kij}\, p_k \, d\Gamma - \int_{\Gamma} S_{kij}\, u_k \, d\Gamma \qquad (2.18)$$

where

$$D_{kij} = \frac{1}{4\pi(1-\nu)r} \{(1-2\nu)\{\delta_{ki}\, r,_j + \delta_{kj}\, r,_i - \delta_{ij}\, r,_k\} + 2\, r,_i\, r,_j\, r,_k\}$$

$$S_{kij} = \frac{2G}{4\pi(1-\nu)r^2} \{2 \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij}\, r,_k + \nu(\delta_{ik}\, r,_j + \delta_{jk}\, r,_i) - 4\, r,_i\, r,_j\, r,_k] + 2\nu(n_i\, r,_j\, r,_k + n_j\, r,_i\, r,_k) + (1-2\nu)(2\, n_k\, r,_i\, r,_j + n_j \delta_{ik} + n_i\, \delta_{jk}) - (1-4\nu)\, n_k\, \delta_{ij}\}$$

The above equations can be used to evaluate the stresses at points within the domain only. If the points are on the boundary a modified approach is adopted as explained in the Appendix A.

CHAPTER 3

FORMULATION FOR CONTACT PROBLEMS

In this chapter contact problem is defined in the first section. After that contact conditions related to static contact, with friction and without friction, are discussed. Contact conditions for rolling contact are presented. Solution procedures for both types of contact problems are given. Two dimensional problems with linear materials are considered.

3.1 Definition of the Contact Problem:

Consider two linearly elastic bodies A and B bounded by cylindrical surfaces. The traces of the boundary surfaces in the x-y plane are the curves. Γ^A and Γ^B respectively. The geometry and all variable quantities are independent of z-coordinate. The problem is thus considered as a plane problem in the x-y system Fig. 3.1. The tractions acting on the boundaries are denoted by t_i and the displacements by u_i. On the boundary Γ_c the two bodies are in contact with each other. The parts of the boundaries corresponding to the contact boundaries Γ_c are in the undeformed state denoted by Γ_c^A and Γ_c^B. $\eta_1 - \eta_2$ is the natural co-ordinate system with η_1 along the normal and η_2 along the tangent of the point on the boundary.

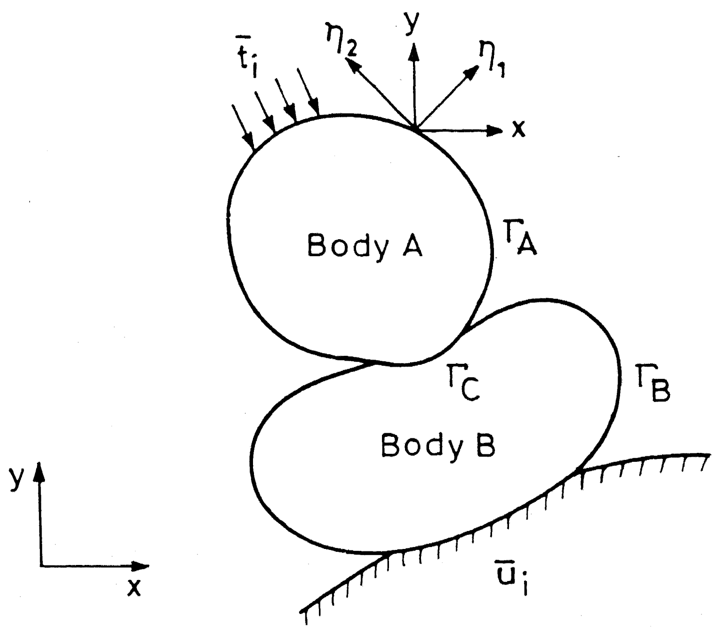


Fig. 3.1 Two bodies in contact along the boundary Γ_C.

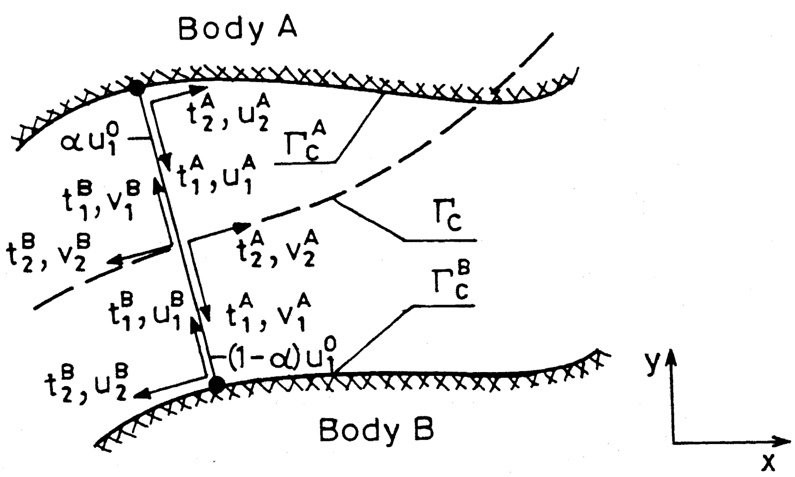


Fig. 3.2 Details of contact region.

3.2 Contact Conditions:

Broadly there are two types of contact problems, viz. static and rolling contact. Contact conditions for both types are given below.

3.2.1 Contact Conditions for Static Contact:

The contact conditions are introduced in natural co-ordinates and later all the equations will be converted to global co-ordinates.

Displacements u_1 and u_2 belong to the undeformed surfaces in natural co-ordinates (Fig. 3.2) whereas v_1 and v_2 are displacements of the same surfaces after contact has been established. 1 and 2 represent the directions of natural co-ordinates. Relations between these two displacements are given below [8] :

$$
\begin{aligned}
v_1^A &= u_1^A - \alpha u_1^o && \text{on } \Gamma_c^A \\
v_1^B &= u_1^B - (1-\alpha)\, u_1^o && \text{on } \Gamma_c^B \\
v_2^A &= u_2^A && \text{on } \Gamma_c^A \\
V_2^B &= u_2^B && \text{on } \Gamma_c^B
\end{aligned}
\qquad (3.1)
$$

where α – a function of the location of contact boundary

u_1^o – the distance in direction 1 between the original positions of the points that are now in contact.

Superscripts A and B indicate the respective bodies.

Static contact conditions can be given for three situations viz.,

(i) No-friction

(ii) Infinite friction

(iii) Finite friction

(i) Contact conditions for no-friction case can be stated as follows on Γ_c

$$
\begin{aligned}
v_1^A + v_1^B &= 0 \\
t_1^A - t_1^B &= 0 \\
t_2^A &= 0 \\
t_2^B &= 0 \\
t_1^A < 0 , \quad t_1^B &< 0
\end{aligned}
\tag{3.2}
$$

(ii) For infinite friction on Γ_c

$$
\begin{aligned}
v_1^A + v_1^B &= 0 \\
v_2^A + v_2^B &= 0 \\
t_1^A - t_1^B &= 0 \\
t_2^A - t_2^B &= 0
\end{aligned}
\tag{3.3}
$$

(iii) For finite friction

$$
\begin{aligned}
\Gamma_c : \quad v_1^A + v_1^B &= 0 \\
t_1^A - t_1^B &= 0 \\
\Gamma_{ca} : \quad v_2^A + v^B{}_2 &= 0 \\
t_2^A - t_2^B &= 0 \\
\text{if } |t_2| &< \mu |t_1|
\end{aligned}
\tag{3.4}
$$

$$\Gamma_{cs} : \begin{array}{l} t_2^k = \pm \mu \, t_1^k \quad k = A, B \\ t_2^A - t_2^B = 0 \\ \text{if } |t_2| \geq \mu \, |t_1| \end{array}$$

where

Γ_c - contact zones
Γ_{ca} - stick zone (No-slip)
Γ_{cs} - slip zone

Sign of μ, coefficient of friction, is decided in such a manner that

$$\text{sign}\,(t_2) \neq \text{sign}\,(v_2^A + v_2^B)$$

The idea is that μ should have that sign which resulted in energy dissipation from the earlier configuration.

Using the relationship in the form of Eqn. 3.1,

$$v_1^A + v_1^B = u_1^A + u_1^B - u_1^o = 0$$

$$\text{i.e.} \quad u_1^A + u_1^B = u_1^o$$

Similarly,

$$v_2^A + v_2^B = u_2^A + u_2^B = 0$$

If we assume the contact length or area is very small and if it is almost parallel to the x-axis, then Eqn. 3.2-3.4 can be rewritten in the global co-ordinates in the following manner:

(i) No-friction:

$$
\begin{aligned}
u_y^A - u_y^B &= u_y^o \\
t_y^A + t_y^B &= 0 \\
t_x^A &= 0 \\
t_x^B &= 0
\end{aligned}
\tag{3.5}
$$

(ii) Infinite friction:

$$
\begin{aligned}
u_y^A - u_y^B &= u_y^o \\
u_x^A - u_x^B &= 0 \\
t_y^A - t_y^B &= 0 \\
t_x^A - t_x^B &= 0
\end{aligned}
\tag{3.6}
$$

(iii) Finite friction:

$$
\begin{aligned}
\Gamma_c :\quad & u_y^A - u_y^B = u_y^o \\
& t_y^A + t_y^B = 0 \\
\Gamma_{ca} :\quad & u_x^A - u_x^B = 0 \\
& t_x^A + t_x^B = 0 \\
& \text{if } |t_x| < \mu |t_y| \\
\Gamma_{cs} :\quad & t_x = \pm \mu t_y \\
& t_x^A + t_x^B = 0 \\
& \text{if } |t_x| \geq \mu |t_y|
\end{aligned}
\tag{3.7}
$$

Subscripts x and y denote the directions x and y of global co-ordinates.

In this work contact conditions given by Eqn. 3.5 - 3.7 are used.

If the contact surface is sufficiently large then all the conditions given in natural co-ordinates are converted to global ones using transformation matrices [8]. On the other hand, the entire problem can be solved in natural co-ordinates itself [8].

3.2.2 Contact Conditions for Rolling Contact:

Unlike in static contact problems, no friction and infinite friction cases are not considered. Only finite friction case is dealt with as it is more practical than the other two conditions.

Analytical approaches [3,12] based on half-plane solutions suggest that the leading edge of the contact zone should have slip region. However experimental studies [16] indicate that there will be indeed slip zone on both the sides with no-slip zone in between. So, the contact conditions for rolling contact are as given below:

$$
\begin{array}{lll}
\Gamma_c : & u_y^A - u_y^B = u_y^o & \\
 & t_y^A + t_y^B = 0 & \\
\Gamma_{cs} : & t_x = \pm\, t_y & \\
 & t_x^A + t_x^B = 0 & \qquad (3.8) \\
 & \text{if } |t_x| \geq \mu |t_y| & \\
\Gamma_{ca} : & t_x^A + t_x^B = 0 & \\
 & u_x^A - u_x^B = \delta_x &
\end{array}
$$

where δ_x is the term that takes care of the rolling contact situation. This 'relative displacement', δ_x, is found out by an iterative procedure, which will be explained in detail later in Sec. 3.4.

3.3 Solution Method of Contact Problems by BEM

Consider two linear elastic bodies A and B in contact (Fig. 3.1). The boundary element equation for these two bodies is

$$c_i \, u_l^i + \int_{\Gamma_m} p^*_{lk} \, u_k \, d\Gamma = \int_{\Gamma_m} u^*_{lk} \, p_k \, d\Gamma \qquad (3.9)$$

$$m = A, B$$

In ordinary stress analysis problems half of the boundary values will be known and the remainings are found out by solving Eqn. 3.9 as explained in the previous chapter. In case of contact problems, neither displacements nor tractions will be known in the contact region and so the number of equations available will be less than the number of unknowns. So, we have to introduce certain equations which govern the contact phenomena and such equations are the contact conditions discussed in the previous section. In this manner number of unknowns and number of equations available can be made equal and the problem is made solvable.

Initially integration of Eqn. 3.9 is carried over the boundaries of A and B independently and the resultant matrix

equation is (refer Eqn. 2.14)

$$\begin{bmatrix} [H_A] & 0 \\ 0 & [H_B] \end{bmatrix} \begin{Bmatrix} u_A \\ u_B \end{Bmatrix} = \begin{bmatrix} [G_A] & 0 \\ 0 & [G_B] \end{bmatrix} \begin{Bmatrix} t_A \\ t_B \end{Bmatrix} \qquad (3.10)$$

Subscripts A and B correspond to bodies A and B respectively.

Variables of contact conditions are rearranged such that in each equation variables of the upper body (A) are expressed in terms of that of lower body (B). According to these relations, in Eqn. 3.10 columns corresponding to the degree of freedom of the nodes of the lower body are added to the corresponding columns of the upper body. By appropriate matrix operations all the unknown variables are brought from the right hand side to the left hand side matrix. After this column transfers for the applied boundary conditions are carried over as explained in Sec. 2.5. The right hand side column $\{t\}$ has only known values and therefore Eqn. 3.10 acquires the form $[A]\ \{X\} = \{B\}$ which can be solved by any standard method available.

3.4 Iteration Procedure to Determine Contact Zones for Finite Friction:

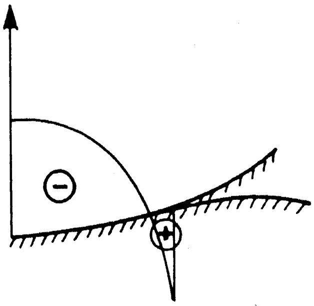
The contact conditions are of non-linear as the contact length will not grow linearly with the applied load so, as any other non-linear problem, to determine contact length iterations have to be done. Iteration procedure for static and rolling contact problems are separately given below.

3.4.1 Static Contact Problems:

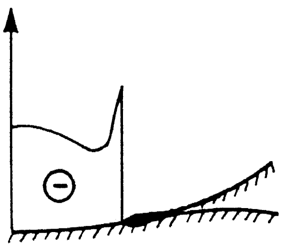
To start with certain portion of the boundary is considered to be in contact and all the necessary contact conditions are implemented. The resulting system of matrices is solved to get the displacements and tractions throughout the boundaries of both the bodies. Since in the contact region there could only be the compressive stress in the direction transverse to the boundaries of contact, this is taken as a reference to decide the contact region, Fig.3.3. In all the assumed contact nodes, there should be compressive stress, otherwise those nodes which have tensile stress will be removed from the contact region. If all the nodes have compressive stress, few more nodes are added to the contact region and the corresponding contact conditions are introduced and the resulting system is solved. This process is repeated till the extreme nodes just begin to have tensile stress. Then with slight rearrangement of the extreme nodes, the points where there are neither compressive nor tensile stress are located. This contact region is the region of contact for the given load and materials, Refer Fig. 3.3.

In determining the 'zero pressure' points, different criteria are followed for constant and linear elements.

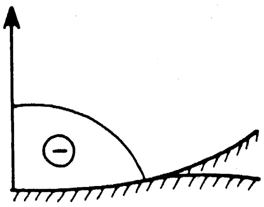
In case of constant element, the extreme pair of contact elements of the contact region on both the bodies are reduced in the length as soon as extreme elements become



Too large contact area
Positive normal stress



Too small contact area
Geometrical incompatibility



Correct contact solution

Fig. 3.3 Three possibilities of solution as results of different choice of contact area.

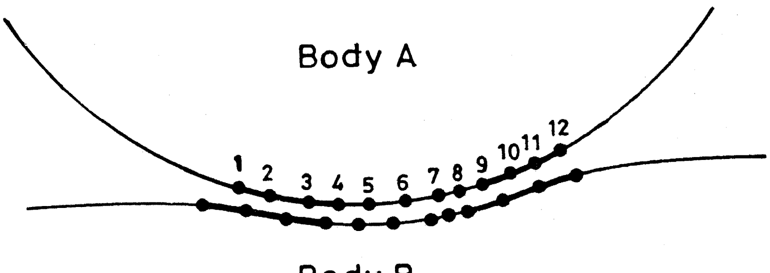


Fig. 3.4 Relative position of the nodes in the contact region in rolling contact problems. (Gap magnified for clarity)

tensile. A better approximation to the contact length is obtained by interpolating using the tractions in y-direction of the extreme pair of elements. With this rearrangement entire solution procedure is carried out again. This procedure is repeated with the updated contact length till a satisfactory value for contact length is obtained.

In case of linear elements, when the extreme nodes acquire tensile stress, interpolation is carried over between extreme pairs of nodes and on both the bodies the lengths of extreme elements alone (unlike in the previous case where a pair of extreme elements together were adjusted) are adjusted, i.e., two elements on the upper body and two elements in the lower body and hence 4 elements are adjusted as against 8 elements in the case of constant elements.

If finite friction is to be considered in the contact region, to start with it is assumed that there is no slip in the contact region at all. After establishing the contact region in the above manner, nodes in which the condition $|t_x| < \mu |t_y|$ is not satisfied are located. For all these nodes contact conditions corresponding to slip region (Eqn. 3.7) are introduced. With these new set of contact conditions matrix system given by Eqn. 3.10 is solved again. Once more all the contact nodes are checked for the violation of the condition $|t_x| < \mu |t_y|$ and if there are such nodes, they are joined with the slipping nodes category and the problem is solved again. This is continued till all the nodes obey the condition $|t_x| \leq \mu |t_y|$.

After that, the nodes immediately outside the contact region are examined for interference, i.e., one body may be piercing the other one. In such case even those nodes are added to the contact region and the problem is solved. On the other hand if there is no interference, the contact length already found out is taken as the correct one and the corresponding traction distributions and displacement fields are used to find out the stress fields.

3.4.2 Rolling Contact Problems:

All the techniques and conditions used for establishing the contact length in case of static contact problems are applicable for rolling contact problems also. However, there is an additional iterative loop which is explained in the following.

Initially the problem is solved as a static contact one. While developing the slip zone in the contact region a relative displacement corresponding the nearest slip node (which is in the leading edge of the contact length) to the no-slip region is introduced in the no-slip region also. In Fig. 3.4 thickened portions are slip regions while the region in between them is the no-slip one. For rolling contact problems the relative displacement corresponding to the node 4 is taken as δ_x, Eqn. 3.8, if 1-2-3-4 is the leading edge. Using Eqn. 3.8 with this δ_x contact problem is

solved. This will give a new relative displacement for node 4. New δ_x is taken to be equal to the new relative displacement for node 4 and the problem is solved again. This iteration procedure continues till the traction distributions in the contact region converge. If there is scope for adding more elements into slip region, iteration continues with new set of slip nodes and again the process explained above is repeated. So, determining δ_x is an iterative loop inside a general iterative loop to determine the slip region of the contact length.

CHAPTER 4

RESULTS AND DISCUSSION

In this chapter results for thick cylinder with internal pressure, static contact and rolling contact are given along with their discussion. All the materials are assumed to be isotropic and linear elastic.

4.1 Thick Cylinder with Internal Pressure:

This is an axisymmetric, two dimensional, plane strain problem. Using symmetry, only one quarter of the thick cylinder is analysed. The boundary is discretised into 52 elements and the elements are of constant type. Analytical solution for this problem is available for the stresses and the displacements in the body. Stresses obtained by BEM and the values of stresses obtained from analytical solutions are given in Table 4.1. The details of the problem are given below:

Inner radius of the cylinder	=	3.3 units
Outer radius of the cylinder	=	7.5 units
Internal pressure	=	100 units
Shear modulus	=	94500 units
Poisson ratio	=	0.1

As can be seen results are in good agreement thus indicating the correctness of the code developed. The discretization is very fine and in fact a model with 26 elements also gave good results.

Table 4.1 : Results for Thick Cylinder

Coordinates		BEM		Analytical	
x	y	σ_r	σ_θ	σ_r	σ_θ
2.6934	2.6934	-69.265	117.51	-69.067	117.08
2.9184	2.9184	-55.463	103.62	-55.270	103.29
3.5034	3.5034	-31.176	79.22	-31.006	79.02
3.8634	3.8634	-21.387	69.39	-21.230	69.25

All the quantities in consistent units.

4.2 Static Contact Problems:

The BEM code is applied to solve static contact problems. A few problems solved by Ma [9] are solved for comparison purposes. They are semi-cylinder over elastic foundation and semi-cylinder over semi-cylinder. Loading conditions are as shown in Fig. 4.1. Constant elements are used. No friction case was solved in [9]. However, the results of this work are for finite friction. For these problems Hertz solutions are available for no-friction state in the contact region. The distribution of contact pressure is shown in Table 4.2. Agreement is satisfactory between the present results and that are given in [9]. If friction in the contact region is considered, then there will be transverse traction in the region. This region can be subdivided into two regions viz. (i) no-slip zone and (ii) slip zone. In no-slip zone μt_x will be less than μt_y and μt_x will be equal to μt_y in slip zone, where t_x and t_y are tractions in the contact region in x and y directions and μ is the sliding coefficient of friction. Though there was not much change in the contact length with respect to μ, lengths of slip and no-slip regions depend on μ. Lower the coefficient of friction larger the slip region for a particular load and vice versa. A limiting value of μ can always be found out such that the entire contact region may be having slip region alone, thus resulting in macro-motion of the body itself.

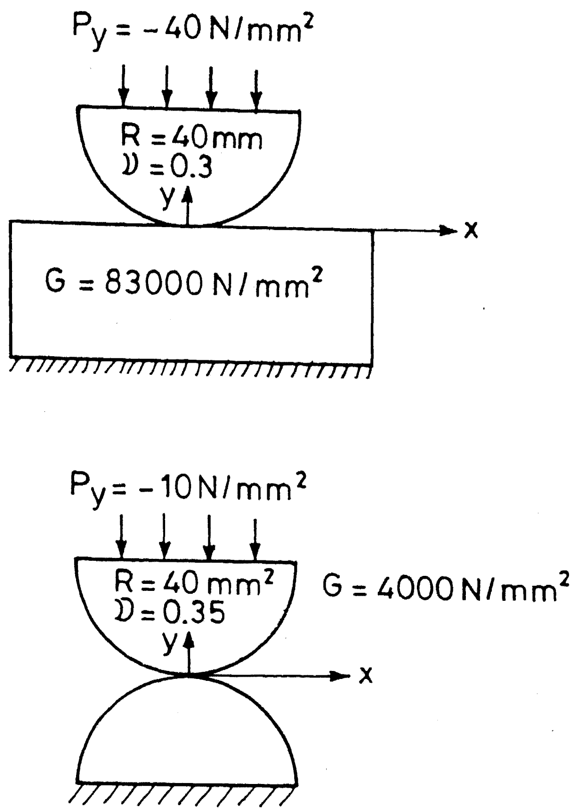
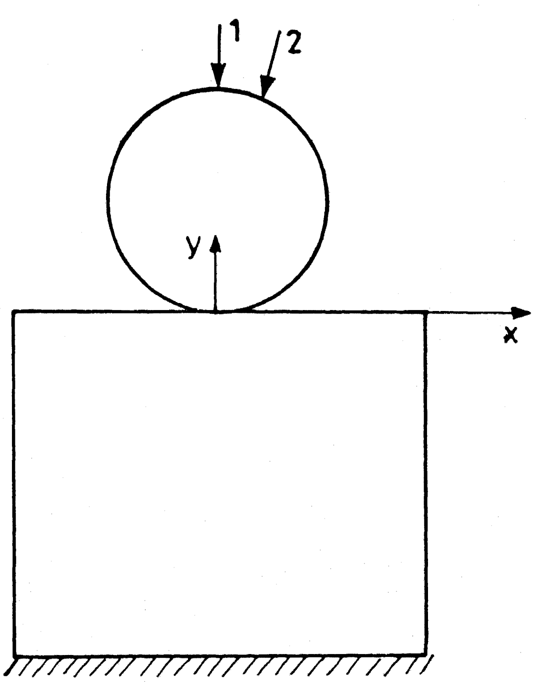


Fig. 4.1 Contact between semi-cylinder and elastic foundation and semi-cylinder and semi-cylinder.



1) Vertical load
2) Inclined load passing through the origin

Fig. 4.2 Cylinder on an elastic foundation.

Table 4.2(a) : Results of Semi-cylinder over Elastic Foundation (Ref. Fig. 4.1)

x(mm)	Contact Pressure (N/mm^2)		
	By Hertz [12]	By Ma [12]	By BEM
1.209	0.0	0.64	-0.3114
1.073	700.0	705.6	673.56
0.875	1156.5	1140.4	1134.3
0.625	1470.1	1484.2	1460.4
0.375	1646.4	1657.8	1631.7
0.125	1727.7	1739.3	1713.6

Table 4.2(b) : Results of Semi-cylinder over Semi-cylinder (Ref. Fig. 4.1)

x (mm)	Contact Pressure (N/mm^2)		
	By Hertz [12]	By Ma [12]	By BEM
1.192	0.0	0.9	6.0
1.621	127.0	128.9	137.56
1.283	198.5	200.3	195.47
0.977	236.1	243.6	239.41
0.672	261.3	259.1	259.96
0.367	274.1	279.1	277.21
0.122	279.3	281.8	282.34

4.2.1 Results with Vertical Load on the Cylinder on Foundation:

A long cylinder over an elastic foundation problem with vertical load (Fig. 4.2) was analysed. The width and height of the foundation are around 4 times and 3 times the radius of the cylinder respectively. Hertz solution is available for no-friction case and Smith and Liu's [2] solution is available for friction cases.

Numerical results by applying BEM technique with constant elements is available in [8] . Fig. 4.3 gives the solutions from the present work and Ref. [8] . It may be noted that t_y distribution of both works are in agreement whereas there is slight discrepancy in the case of t_x distribution. In Ref. [8] loading is done in several increments whereas in this work total load is applied in one step because of the high amount of computer time involved. Nevertheless the traction distributions thus obtained are sufficiently accurate to give a satisfactory stresses as will be demonstrated in the later part of this section.

Distributions of contact pressure (t_y) and t_x in the contact region for a cylinder of radius 72 mm is shown in Fig. 4.4. Linear elements are used for modelling. This class of problems does not show much difference between friction and no-friction cases, regarding t_y distribution whereas t_x distribution changes with coefficient of friction, because of the change in slip and no-slip zones.

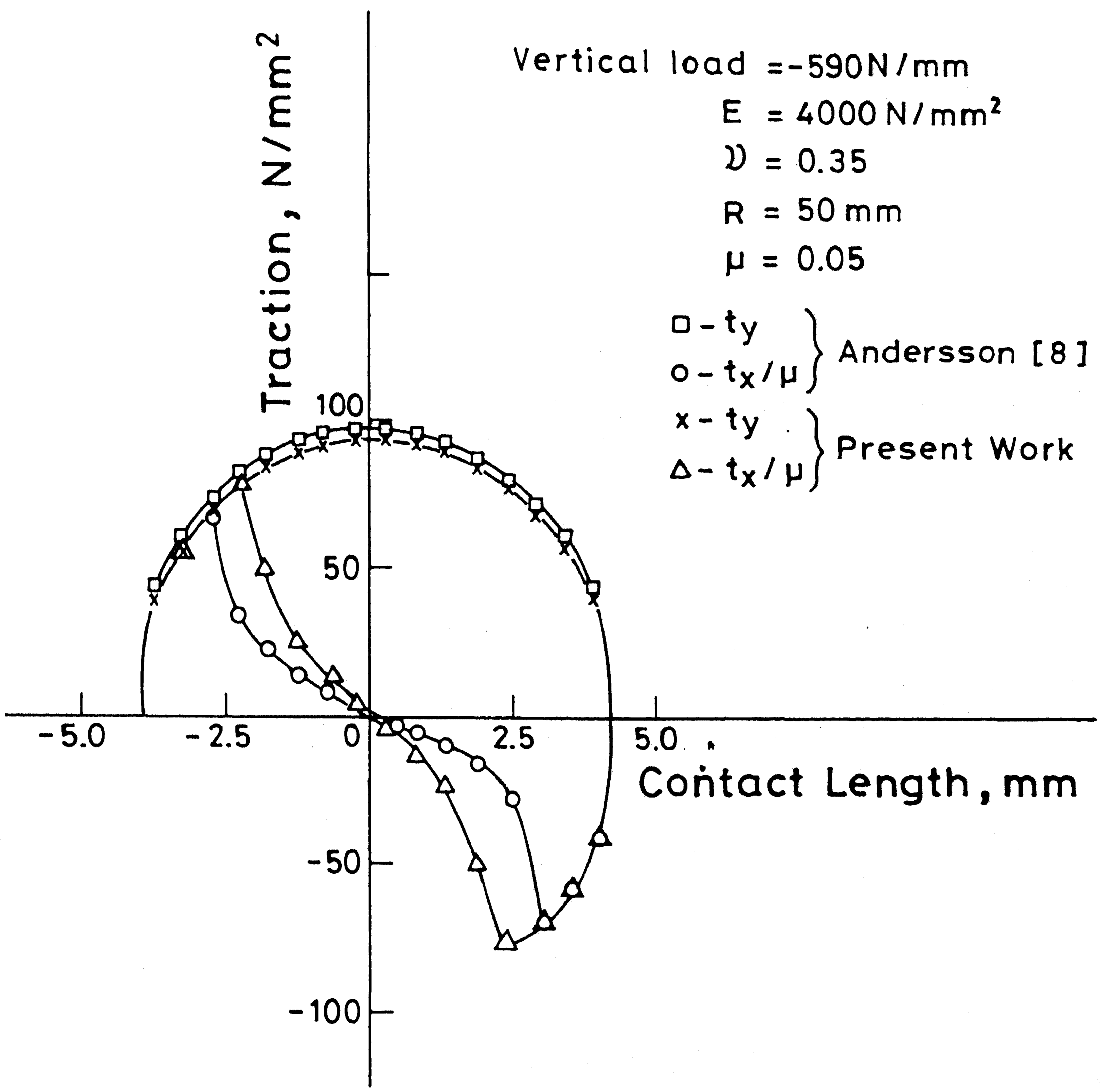


Fig. 4.3 Cylinder on elastic foundation, t_y and t_x for cylinder in the contact zone.

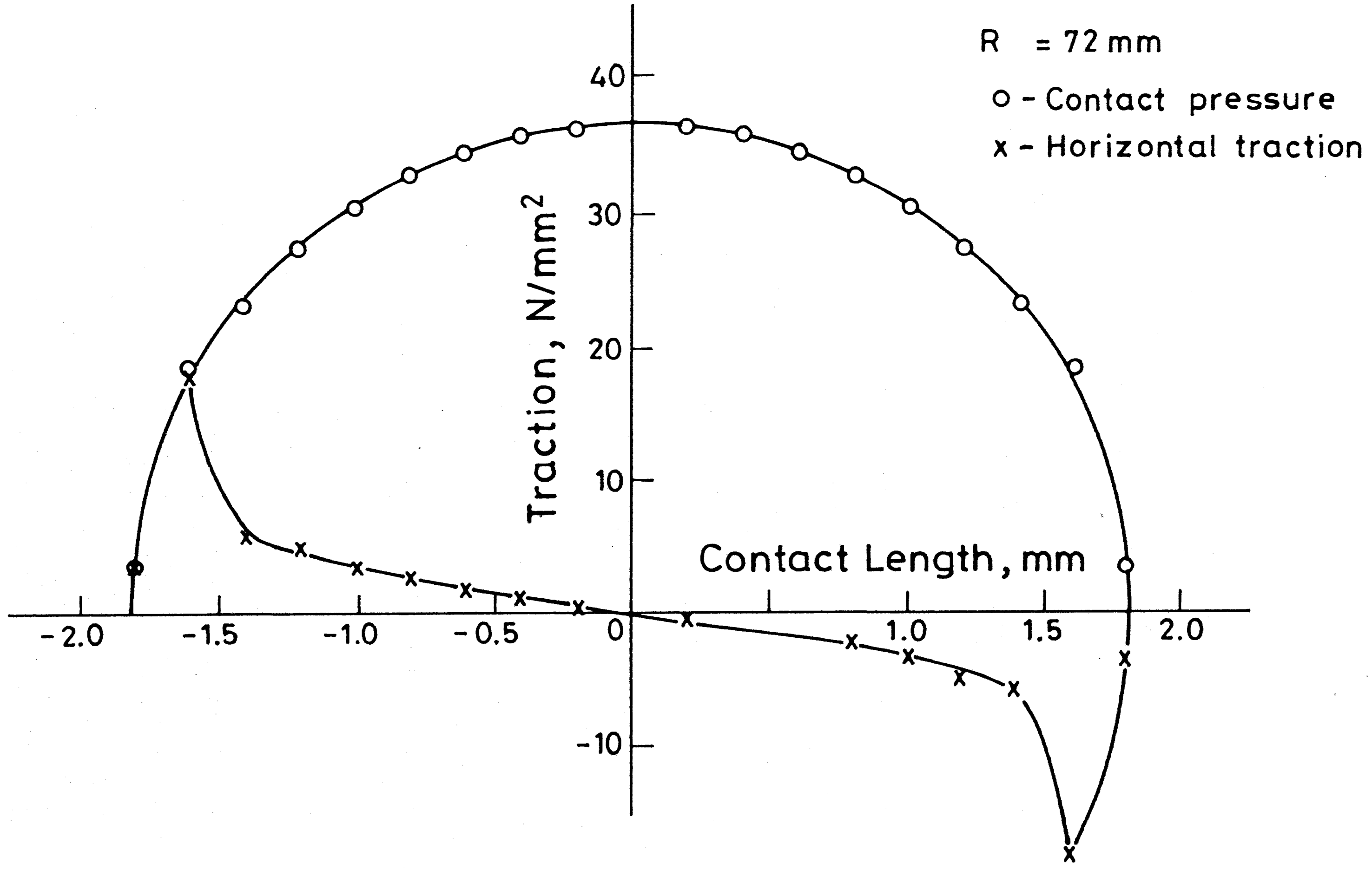


Fig. 4.4 Traction-t_x/μ and traction-y distribution (plane strain) in the contact region of a cylinder over elastic foundation with $E = 4000$ MPa, Vertical load $= -150.61$ N/mm, $\nu = 0.35$ and $\mu = 0.01$.

For a vertical load on the cylinder, Fig. 4.2, variation of stresses along the vertical diameter is shown in Fig. 4.5. Along with this plot solution by Smith and Liu [2] is also shown. While the solutions are matching for σ_y and σ_{xy} around the contact region, marked variation is seen in the case of σ_x near the contact region itself. For some points [2] gives σ_x as compressive stress whereas BEM solutions for these points give tensile stress values. This is due to the fact that Smith and Liu [2] solution was obtained from a basic half-plane solution. The BEM solutions can be checked in the following manner. Away from the contact region this solution will be similar to a cylinder loaded at diametrically opposite points [1]. Assuming each of the loads produces a simple radial stress distribution, it can be found out what forces should be applied at the circumference of the disk in order to obtain such a stress distribution. Stresses on the boundary should be nullified by applying equal and opposite stresses, if the boundary is traction free. So, away from the point loads this problem and the contact problem are one and the same according to St. Venant's principle. As can be seen from Fig. 4.5 away from the contact region BEM solutions and Hertz solution in [1] are matching well.

A similar superposition procedure was carried out with elliptically varying load being applied instead of the point loads on the cylinder. Half-plane solutions for this kind of load is available in [2]. However, unlike in the point load

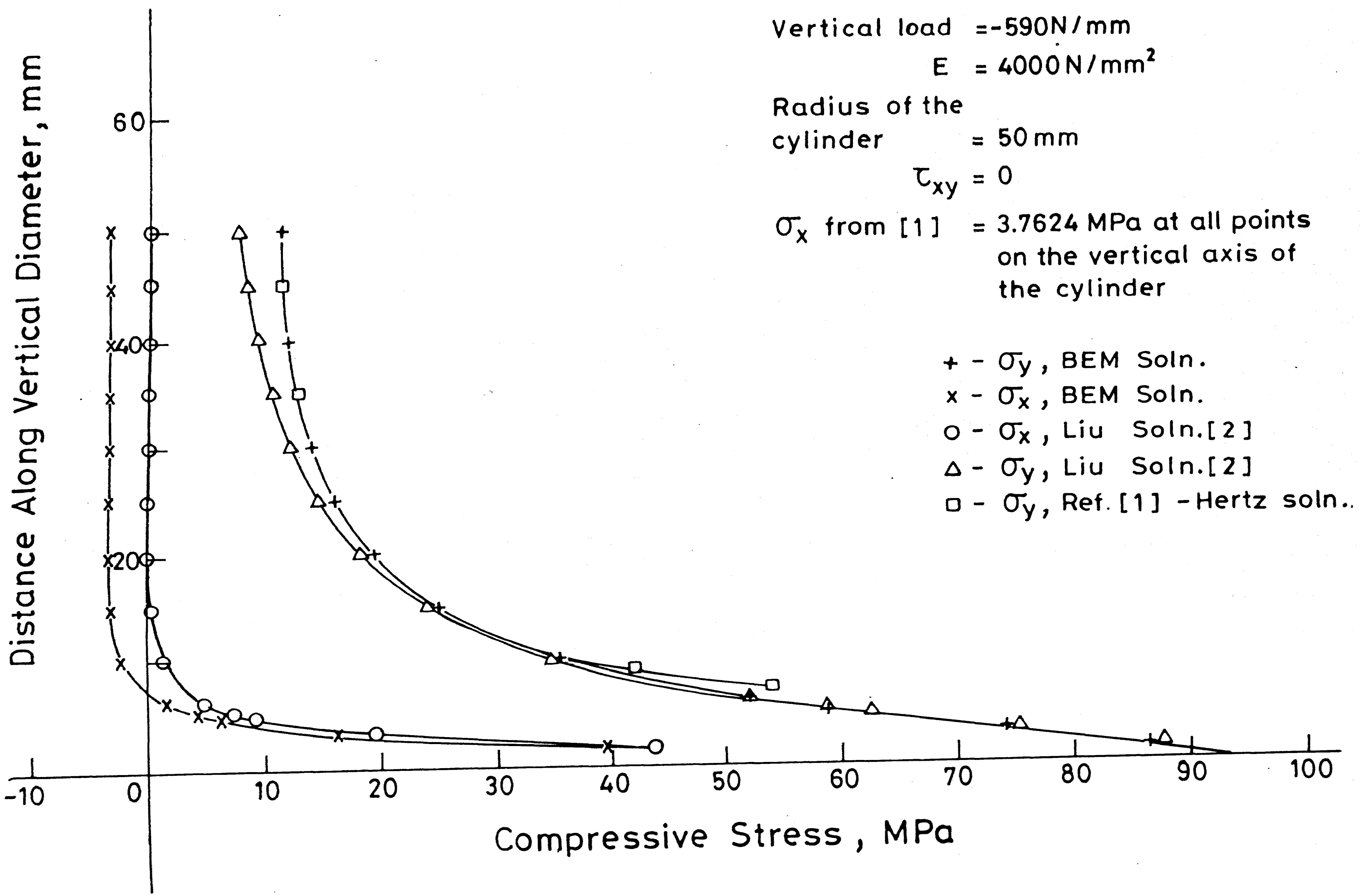


Fig 4.5 Comparison of the solutions from different sources for a cylinder

case, here the stress field to be applied on the cylinder to make it traction free on the boundary is not uniform. So, solution for this stress field is obtained from BEM itself and used for superposition. The final solution of σ_y and σ_{xy} were in agreement with Smith and Liu near the contact region and with Hertz [1] away from the contact region. The final solution of σ_x was in agreement with BEM solutions at all the points. However, the results are not presented as this problem is almost similar to the point load case.

4.2.2 Results of Inclined Load on the Cylinder:

When compared with the results of vertical load, there is marked change in the t_x distribution (Fig.4.6). This is because the inclined load that introduces a horizontal force and contact region is the only area where the reaction for this horizontal force will exist. In this case t_y distribution is also changed. It is skewed in such a manner that the resultant vertical load can counteract the moment introduced by the horizontal load.

Stress distribution curves for the slightly inclined load (Fig. 4.2) are shown in Fig. 4.7. It may be noted that this load does not introduce any moment. Stress distributions due to vertical load 150.6 N/mm are also shown in the same figure for the sake of comparison.

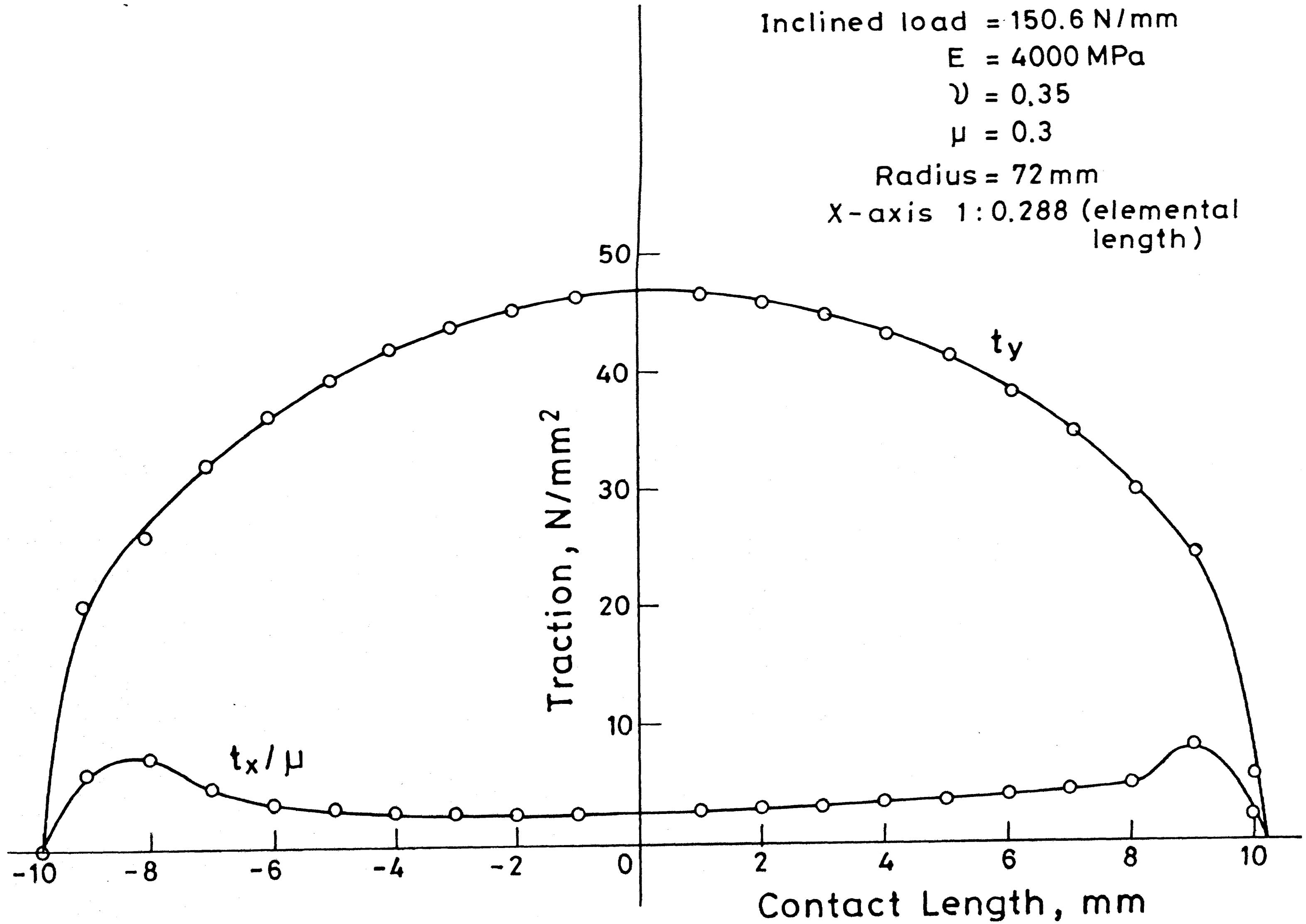


Fig. 4.6 t_x and t_y distributions on the contact region of a cylinder loaded with an inclined load.

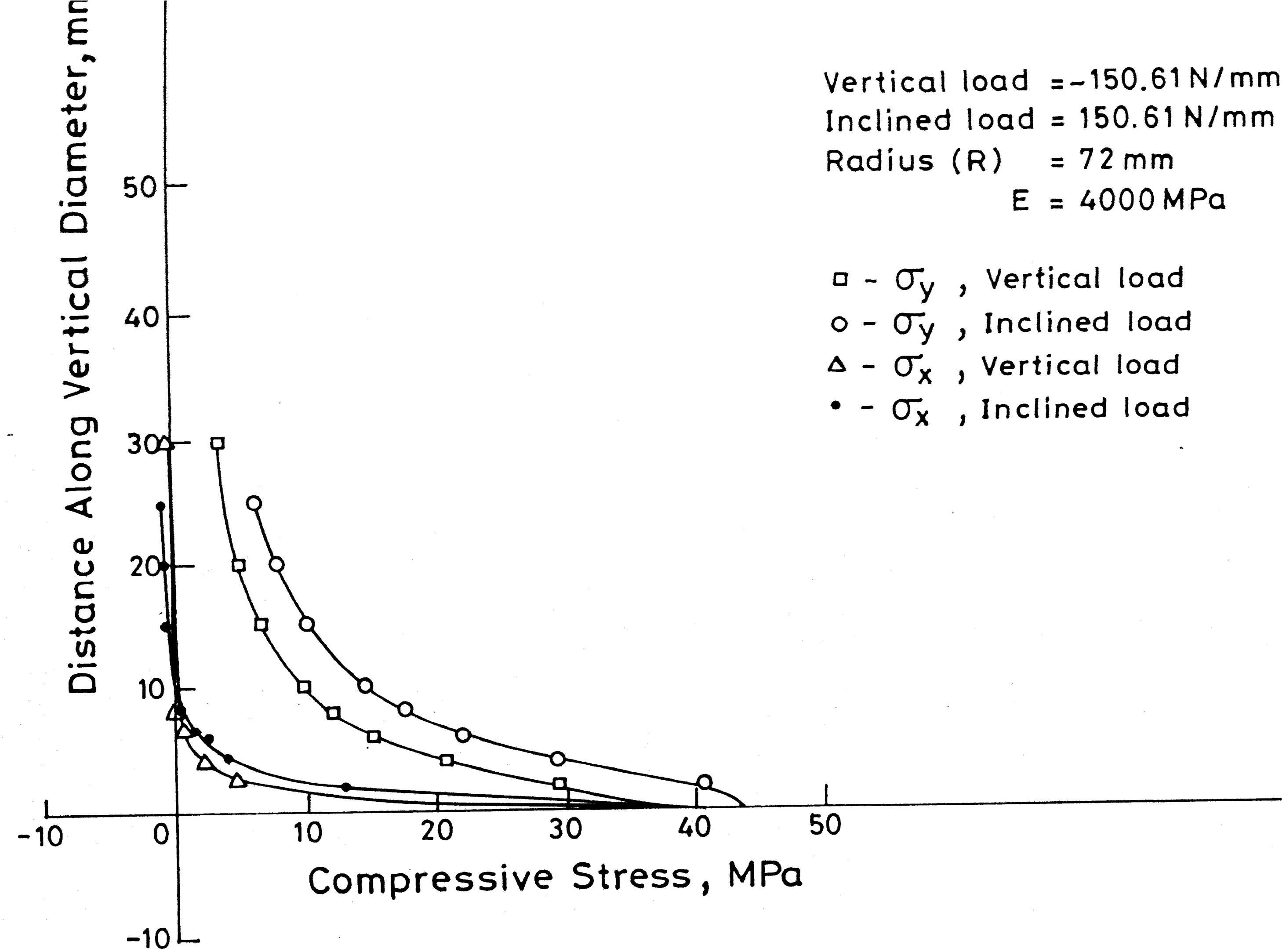


Fig. 4.7 Stress along the central vertical plane of the cylinder in contact for vertical and inclined loads.

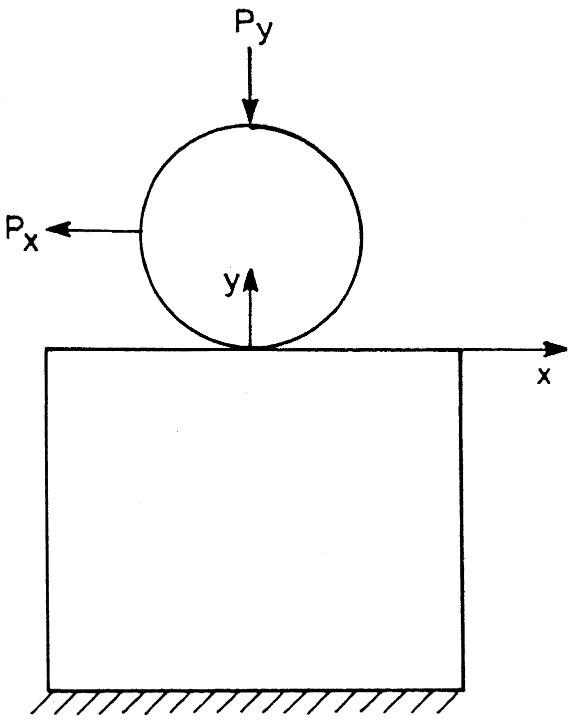
4.3 Dynamic Contact or Rolling Contact Problems:

A static contact problem itself is a highly nonlinear one and many such problems have to be solved with short time intervals to simulate rolling contact. So, for dynamic problems calculations involved are very complex. Computational work is reduced very much if only slow rolling is considered. It is assumed that contact conditions, region etc. are not affected by the value of rolling speed. Contact conditions as discussed in Section 3.2.2 are used.

In order to study the effect of free rolling of a cylinder on an elastic foundation, loads are applied as shown in Fig. 4.8. For each normal load, various values of horizontal loads are applied. Horizontal loads are of small magnitudes with approximately 1/5000 to 1/100 of the vertical load P_x and P_y correspond to the applied vertical and horizontal tractions whereas t_x and t_y correspond to the traction distributions at the contact region.

4.3.1 Results for the Case of Same Material for Cylinder and Foundation:

When same material is used for both cylinder and foundation the distribution of t_y and t_x are as shown in Fig. 4.9 for one particular applied P_y and P_x. It can be noted that t_y distribution in the contact region is skewed to the left. This is expected because of the moment introduced by the horizontal



P_y - Applied vertical load / Elemental length
P_x - Applied horizontal load / Elemental length

Fig. 4.8 Loading conditions in rolling cylinder.

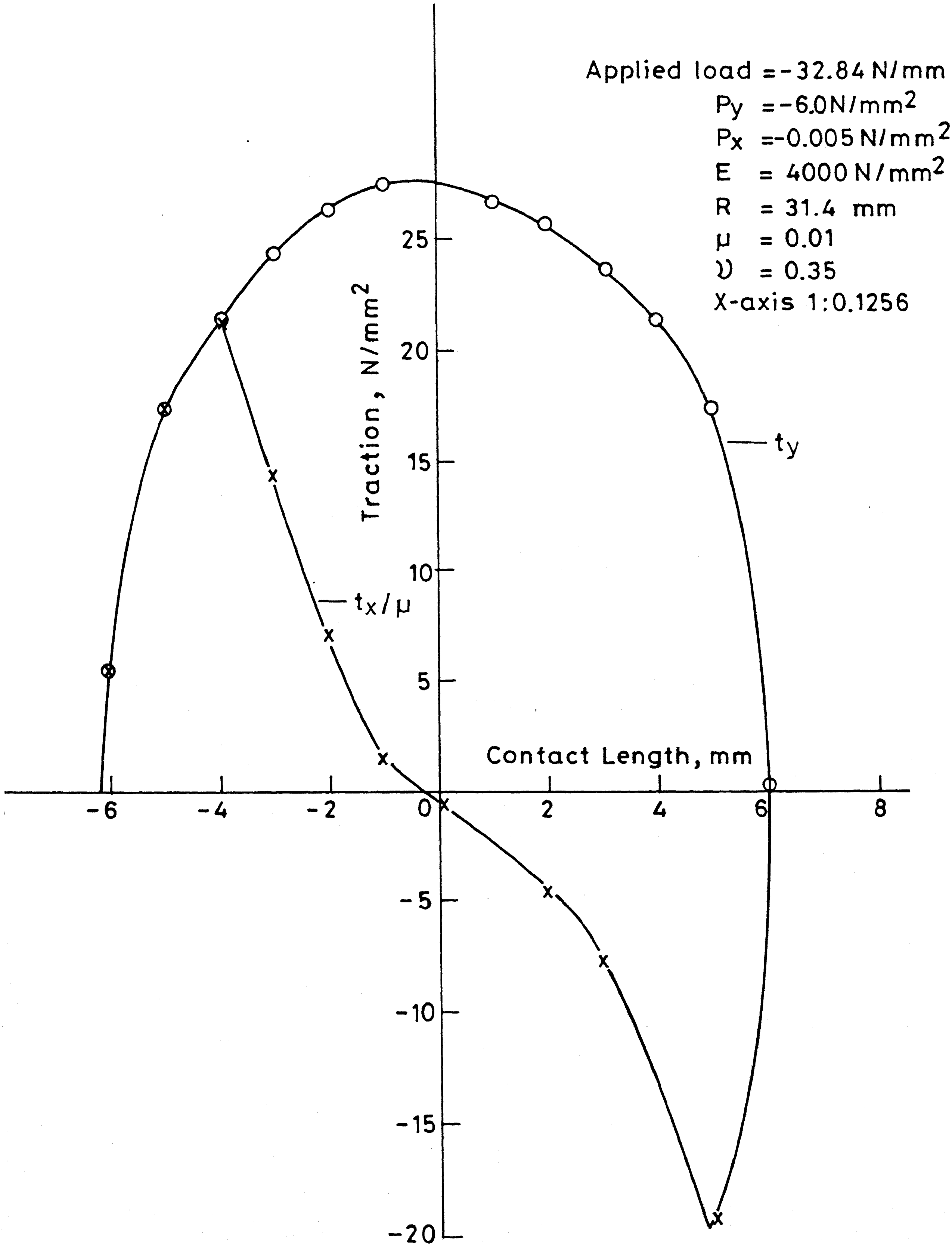


Fig. 4.9 Traction distributions for rolling contact situations when both cylinder and the elastic foundation are of same material.

load. The skewed distribution of t_y introduces a counter moment which can be calculated by using the t_y distribution itself.

Stresses for points on the vertical axis, Fig. 4.10, are given for two horizontal loads and for one vertical load. Due to the asymmetry in the stress field σ_y, when compared with the solutions for vertical load alone, has lower values for all the corresponding points. As P_x increases the point where maximum σ_y occurs also keeps moving to the left. For the same reason τ_{xy}, which was having zero values in the case of vertical load only along the line of symmetry, increases for the corresponding points.

σ_x increases with increase in the applied horizontal load because of the nature of σ_x stress field (Fig. 4.11). On any given horizontal plane the trend of σ_x distribution is as shown in Fig. 4.11. As shown in the graph, this distribution when skews to the left value of σ_x at the centre-line increases with applied horizontal load. (Though the graphs in Fig. 4.11 are for Teflon cylinder and steel foundation, same trend is noted in the present case also).

4.3.2 Results for the Case of Different Materials for Cylinder and Foundation:

The above study was also conducted for two other combinations viz. (i) PMMA (plexiglass) cylinder on steel foundation and (ii) Teflon cylinder on steel foundation.

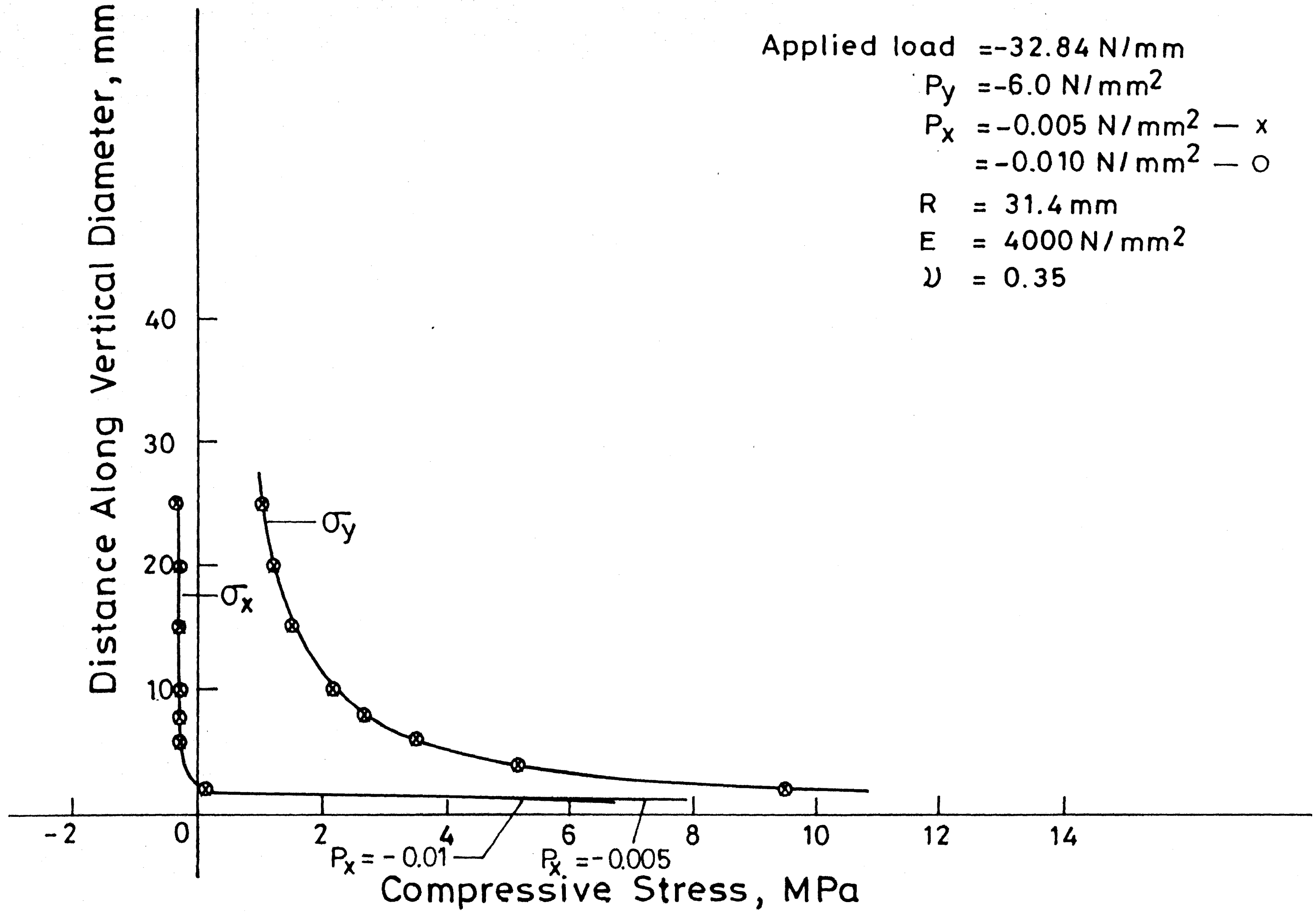


Fig. 4.10 Stress distribution along the central vertical plane of the cylinder (R = 31.4 mm) in rolling contact. Elastic foundation is also of same material (plane strain)

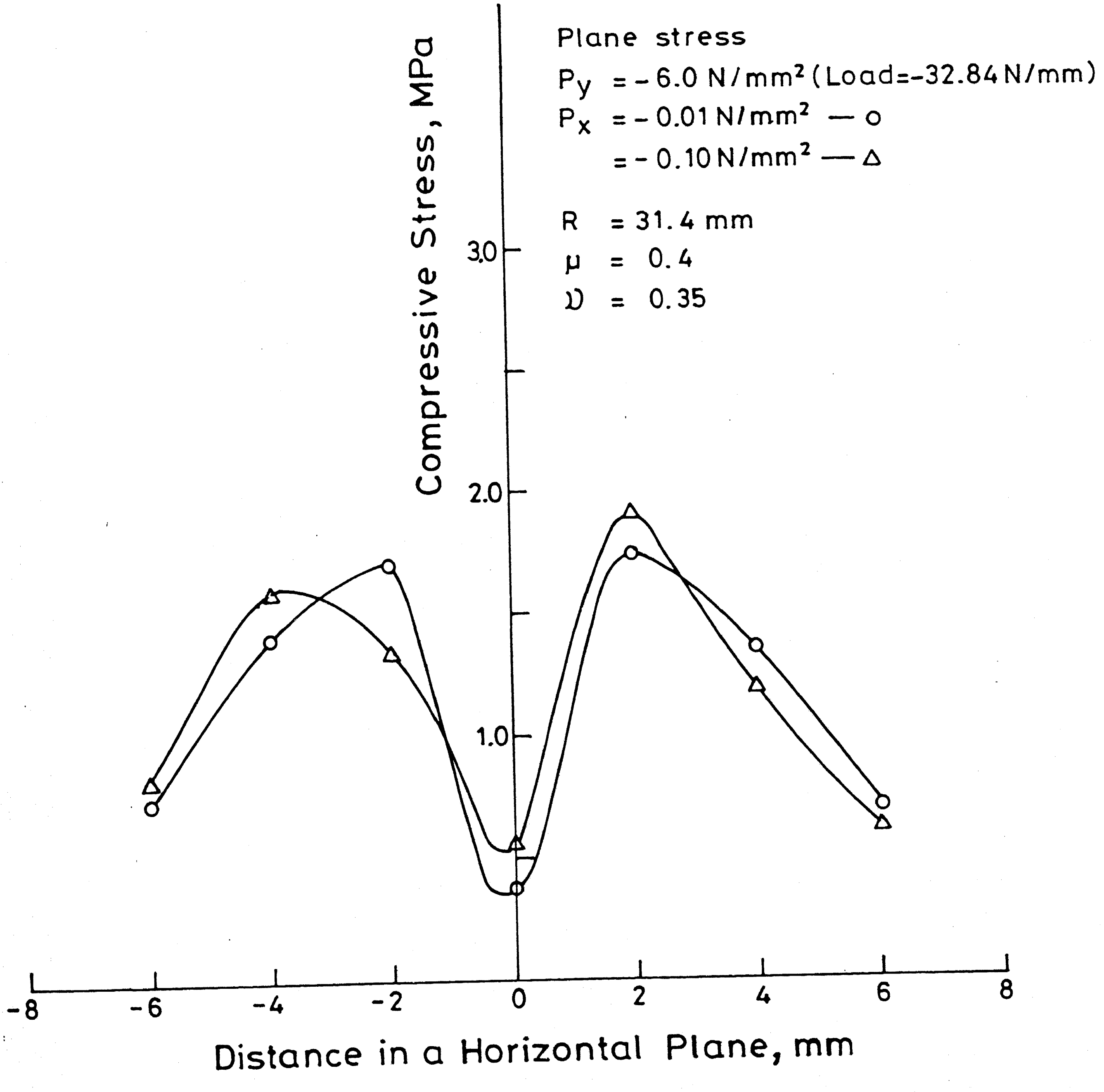


Fig. 4.11 σ_x distribution on a plane y = 2 mm of a teflon cylinder rolling on steel foundation.

It is noteworthy of attention to see the difference in the pattern of t_x distribution, Fig. 4.12, between this case and the former one, Fig. 4.9, where both the bodies are made of same material. A parametric study was carried out to see when this distribution changes its pattern from the one shown in Fig. 4.9 to the present one. The modulus of elasticity for the cylinder was kept constant and that of foundation was increased gradually. At a ratio of around 1.1 almost all the t_x in the contact region vanishes and afterwards the present trend sets in. Actually the source of t_x is the interaction between displacement fields of the cylinder and the foundation. As the elasticity modulus of the foundation is raised there is more deformation in the cylinder and hence the change in traction distribution pattern. However, t_y distribution (Fig. 4.13) retains the same trend as that for cylinder and foundation made of same material because it is mostly governed by the vertical load.

Stresses are calculated along the vertical axis of the cylinder for both the combinations for a few vertical loads with different horizontal loads. σ_x and σ_y stresses are plotted in Fig. 4.14 with vertical load as 32.84 N/mm and horizontal loads as 0.0547 N/mm and 0.02737 N/mm for Teflon and steel combination. Results of a few other cases are given in Table to 4.3 to 4.7.

In order to determine the contribution of micro-slip towards the overall rolling friction coefficient, we have to find the horizontal load necessary for a steady rolling. The

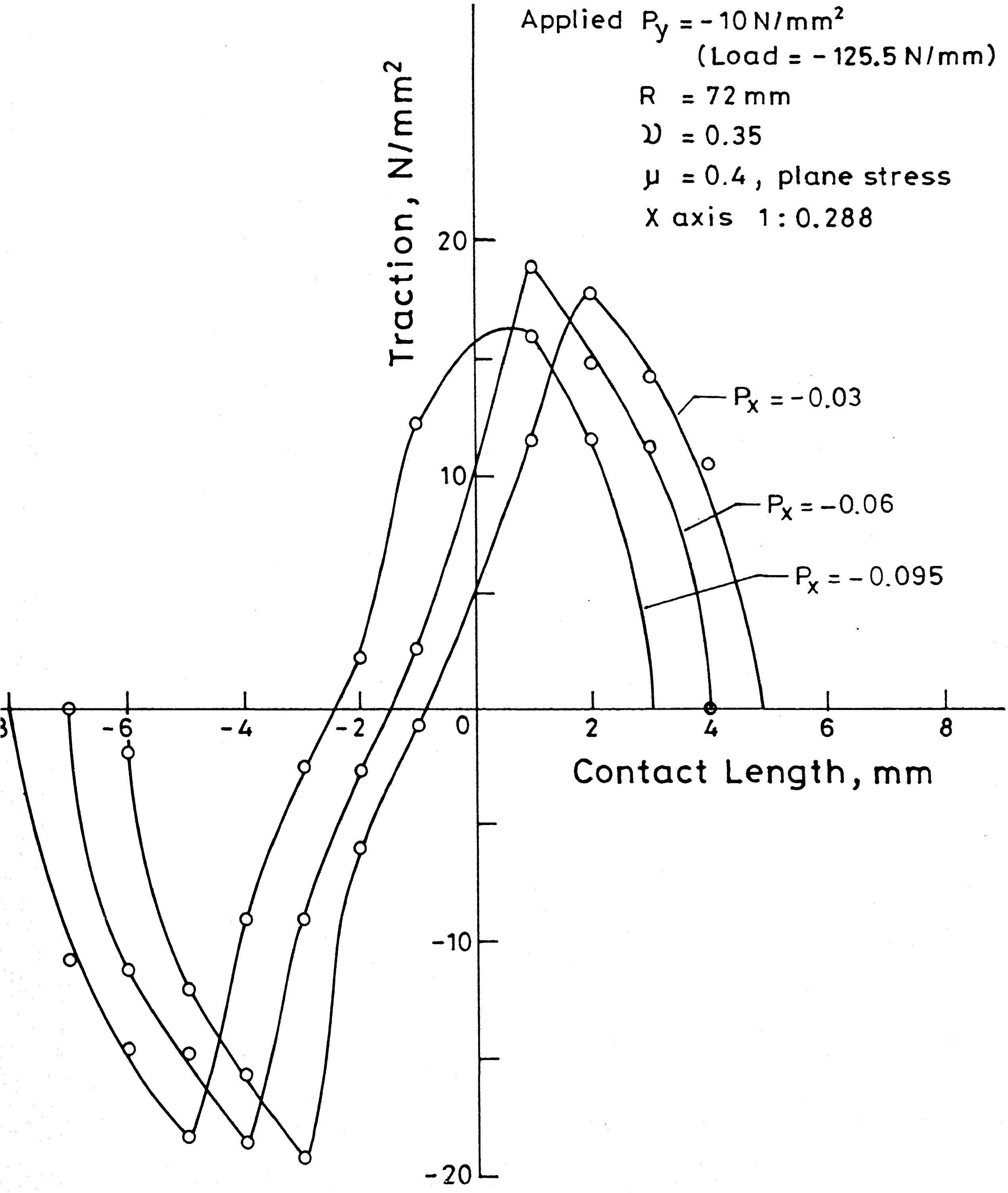


Fig. 4.12 Traction along x direction in the contact region for various horizontal loads acting on a PMMA cylinder rolling on a steel base.

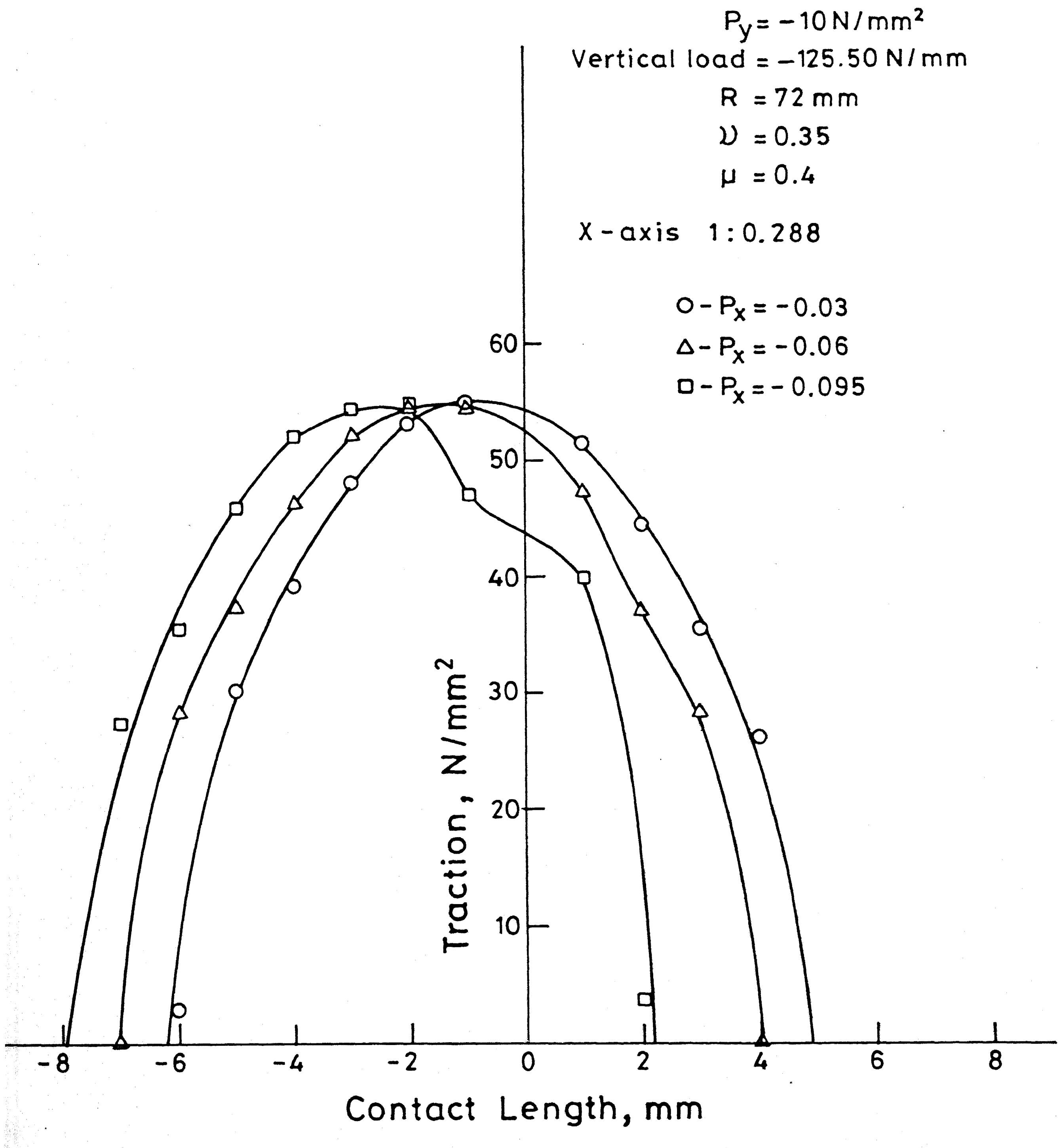


Fig. 4.13 Traction along y-direction for various horizontal loads acting on a PMMA cylinder rolling on a steel foundation.

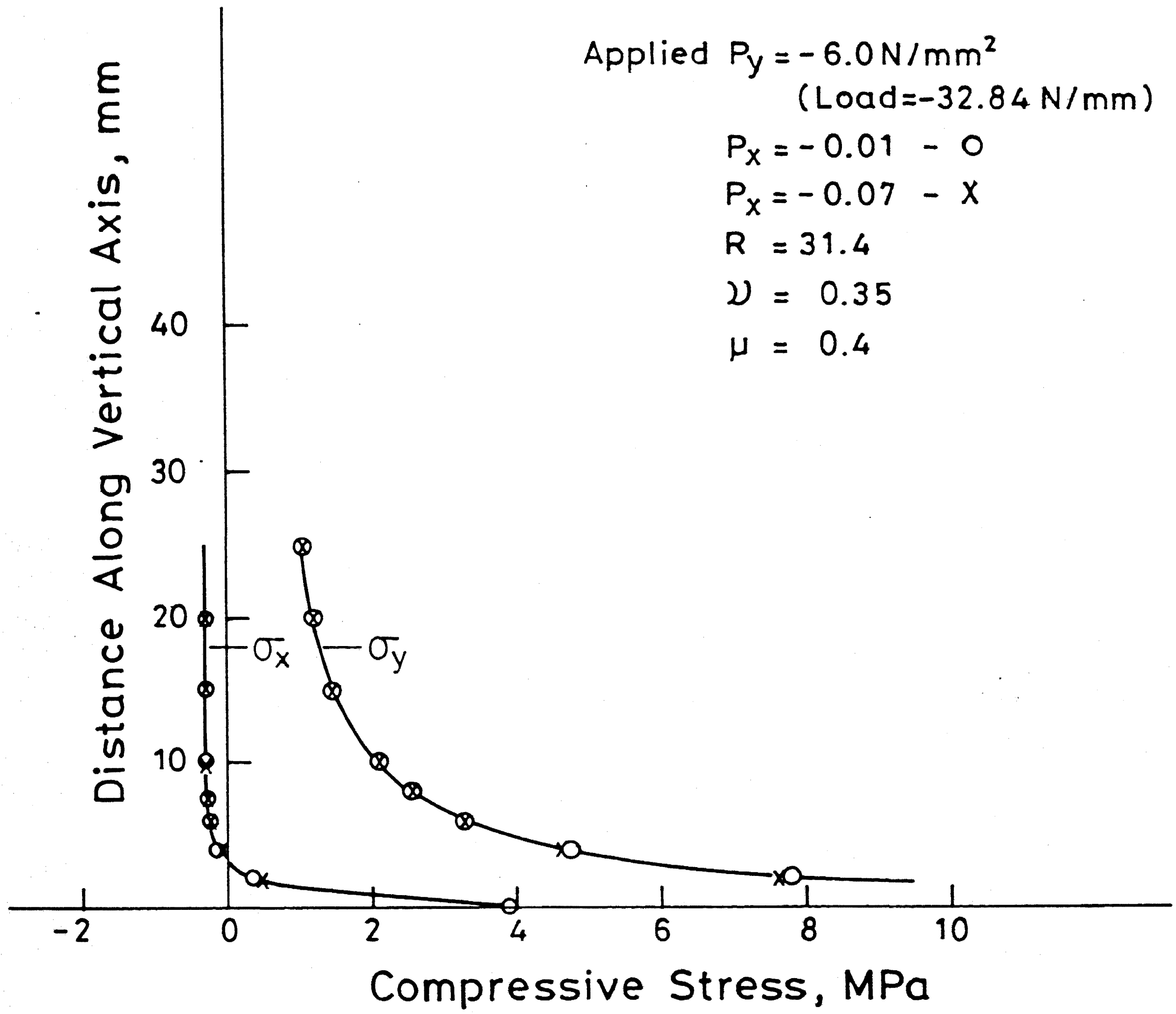


Fig. 4.14 Comparison of stress fields for two different horizontal loads in a Teflon cylinder on steel foundation.

resultant moment about the centre of the cylinder due to each loading condition is calculated and for a particular. Vertical load when value of horizontal load is small this resultant moment will act such that a free rolling cylinder will decelerate . As the horizontal load is increased resultant moment gradually changes its sign so that the body starts accelerating [16]. When the resultant moment is zero, that situation corresponds to steady rolling. For one particular case of PMMA and steel combination, resultant moment is plotted against the applied horizontal load in Fig.4.15. Resultant moment is calculated by finding the horizontal and vertical reactions in the contact region and taking moments about the centre of the cylinder.

The position of the resultant vertical force in the contact region shows very smooth behaviour, Fig. 4.15, with applied horizontal loads for all the cases, whereas the resultant moment is not having a uniform trend. These trends can be explained as follows. From Fig. 4.12 it can be seen that horizontal reaction will be the algebraic addition of positive and negative areas of a t_x distribution curve. While a slight different distribution may also be accepted as a correct one, horizontal force in the contact region may vary considerably. In the process error gets accumulated in horizontal reaction. So, being a small value, it is very likely that the value of resultant moment will have large error and this is suspected to be the reason for an unstable behaviour of resultant moment.

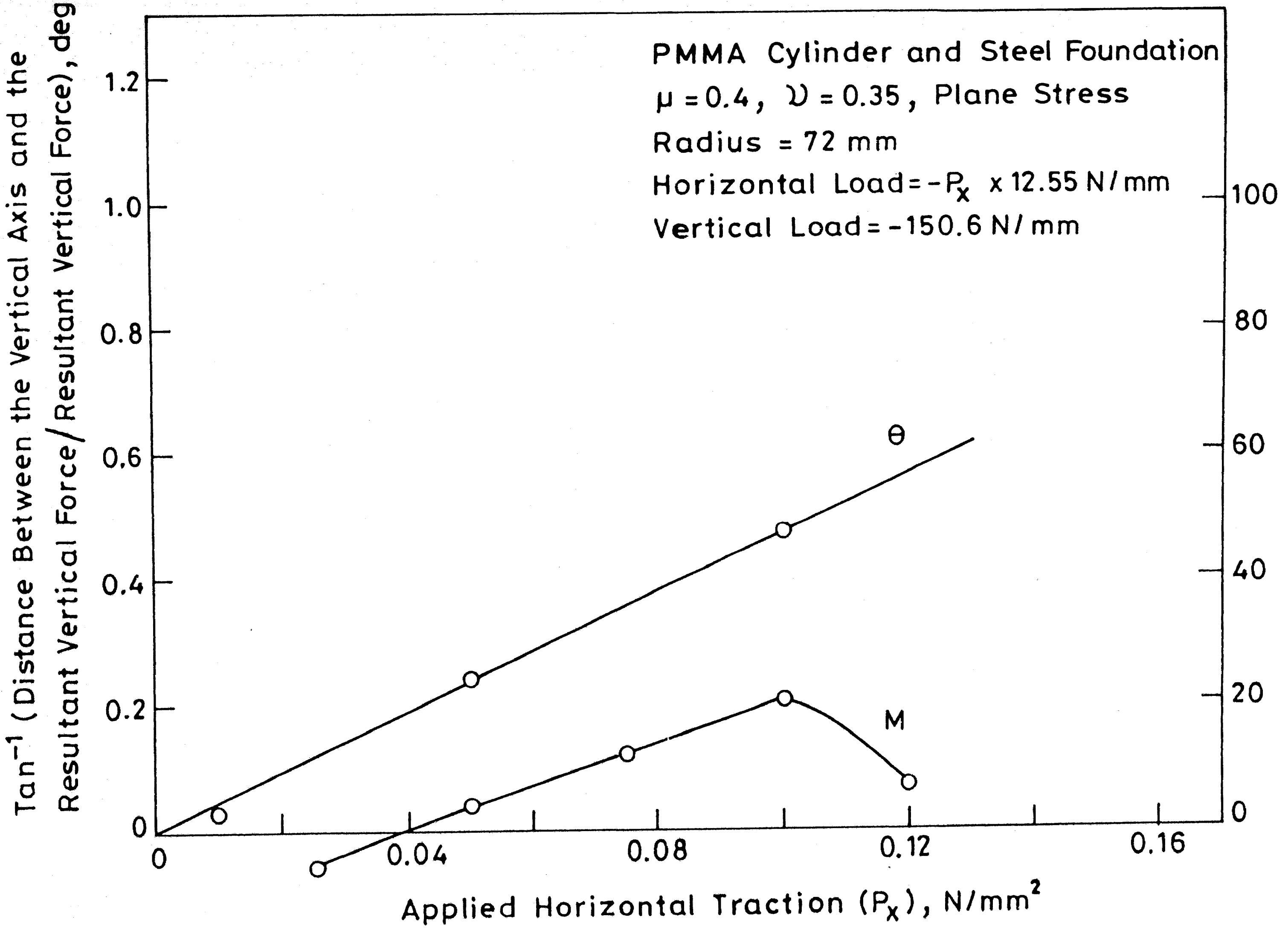


Fig. 4.15 Variation of resultant moment and the position of the resultant vertical force in the contact region against applied horizontal load for PMMA cylinder on steel foundation.

From Fig. 4.15 rolling friction coefficient, μ_R, can be calculated if the horizontal load for zero resultant moment is divided by the applied vertical load and for PMMA cylinder on steel foundation $\mu_R = 0.0033$ from the BEM results. This value is much higher when the usual value is around 0.001 . Improvement in μ_R can be expected if the numerical instability in the calculation of horizontal reaction force at the contact region can be avoided.

Table 4.3 : PMMA Cylinder on Steel Foundation (Plane Stress) : Vertical load = 75.3 N/mm. Traction Distributions on the Cylinder

$P_y = -6$ N/mm^2, Radius = 72 mm, $\mu = 0.4$, $\nu = 0.35$, $E_{PMMA} = 4.57$ GPa

x Co-ordinates mm	$P_x = -0.01$ N/mm^2		$P_x = 0.04$ N/mm^2		$P_x = -0.0625$ N/mm^2	
	t_x	t_y	t_x	t_y	t_x	t_y
-2.016					1.7154	-4.2884
-1.728			1.1335	-2.8337	-9.5123	23.782
-1.44	2.9843	-7.4607	-9.7107	24.277	-12.863	32.159
-1.152	-9.0981	22.745	-13.052	32.631	-10.075	39.398
-0.864	-12.380	30.949	- 9.4480	39.693	- 1.6597	42.308
-0.576	-13.082	38.746	- 1.5583	42.588	2.4917	42.431
-0.288	- 2.2721	42.899	6.4792	42.026	14.365	38.373
0.288	9.7643	39.853	12.774	31.934	9.3520	23.380
0.576	12.799	31.997	8.877	22.193	-2.9500	-7.3751
0.864	9.5513	23.878	-3.3792	-8.4480		
1.152	-1.5785	-3.9462				

Table 4.4 : PMMA Cylinder on Steel Foundation (Plane stress). Vertical Load = 150.61 N/mm. Traction Distributions on the Foundation

$P_y = -12$ N/mm^2, Radius = 72 mm, $\mu = 0.4$, $\nu = 0.35$, $E_{PMMA} = 4.57$ GP

x-co-ordinates mm	$P_x = -0.01$ N/mm^2	
	t_x	t_y
1.692	-0.73452	-1.8363
1.440	-8.5447	-21.362
1.152	-14.757	-36.892
0.864	-18.414	-46.036
0.576	-17.986	-53.941
0.288	- 5.2529	-58.538
-0.288	3.6150	-59.217
-0.576	11.819	-56.535
-0.864	20.155	-50.382
-1.152	16.517	-41.293
-1.44	15.245	-33.111
-1.8	-0.031327	- 0.078318

Continued.......

Table 4.4 (Continued):

x-coordinates mm	$P_x = -0.025\ N/mm^2$	
	t_x	t_y
1.566	-0.52807	-1.3202
1.2528	-12.519	-31.297
0.9396	-16.505	-41.263
0.6264	-20.375	-50.937
0.3132	-8.8185	-57.582
-0.3132	1.9841	-59.647
-0.6264	10.256	-57.095
-0.9396	20.296	-50.741
-1.2528	16.342	-40.854
-1.566	12.34	-30.850
-1.8792	-0.05596	0.1399

Continued.......

Table 4.4 (Continued):

x-coordinates mm	$P_x = -0.05$ N/mm^2	
	t_x	t_y
1.4472	0.61635	1.5409
1.152	-11.767	-29.417
0.8928	-15.248	-38.12
0.576	-19.264	-48.159
0.288	-14.099	-55.724
-0.288	- 0.59893	-59.974
-0.576	4.9905	-58.833
-0.864	13.647	-55.7
-1.152	19.553	-48.882
-1.44	15.817	-39.543
-1.728	12.052	-30.131
-2.0304	- 0.30543	+ 0.76358

x-coordinates	$P_x = -0.1$ N/mm^2	
	t_x	t_y
1.152	- 0.37684	-0.94211
0.576	-17.911	-44.777
-0.576	- 4.9532	-60.322
-1.152	13.727	-55.452
-1.728	17.486	-43.714
-2.304	1.2855	- 3.2138

Contd.......

Table 4.4 (Continued):

x-coordinates mm	$P_x = -0.12\ N/mm^2$	
	t_x	t_y
1.008	0.079619	0.19905
0.864	-8.2233	-20.558
0.576	-13.719	-34.298
0.288	-18.222	-45.555
-0.288	-12.717	-56.984
-0.576	- 2.9467	-59.396
-0.864	1.9699	-59.038
-1.152	6.8797	-57.534
-1.44	18.91	-52.35
-1.728	17.464	-43.66
-2.016	13.535	-33.837
-2.3040	6.5141	-16.285
-2.4408	0.29756	- 0.74390

x-coordinates	$P_x = -0.14\ N/mm^2$	
	t_x	t_y
0.8928	-0.35573	-0.88931
0.576	-12.416	-31.040
0.288	-16.592	-41.480
-0.288	-16.964	-55.35
-0.576	- 4.9401	-59.126
-0.864	- 0.17571	-59.391
-1.152	4.9587	-58.477
-1.44	13.267	-55.070
-1.728	19.215	-48.037
-2.016	15.331	-38.327
-2.304	11.307	-28.268
-2.5848	-1.1866	2.9665

Table 4.5 : PMMA Cylinder on Steel Foundation (Plane Stress) Vertical Load = 125.5 N/mm. Traction distributions on the foundation

x-coordinates on foundation (mm)	$P_x = -0.01\ N/mm^2$	
	t_x	t_y
1.512	0.12969	0.32423
1.152	-13.005	-32.512
0.864	-15.826	-39.564
0.516	-19.343	-48.357
0.288	- 5.1569	-53.562
-0.288	2.8598	-54.261
-0.576	13.075	-50.948
-0.864	17.683	-44.208
-1.152	13.972	-34.930
-1.44	9.5431	-23.858
-1.656	-0.56135	1.4034

x-coordinates on foundation (mm)	$P_x = -0.045\ N/mm^2$	
	t_x	t_y
1.2672	0.20488	0.5122
0.9504	-11.833	-29.583
0.6336	-15.576	-38.94
0.3168	-16.792	-49.097
-0.3168	- 0.044009	-55.066
-0.6336	5.2803	-53.566
-0.9504	17.779	-48.637
-1.2672	15.733	-39.332
-1.584	11.897	-29.742
-1.9008	-0.01165	0.027913

Continued......

Table 4.5 (Continued):

x-coordinates on foundation (mm)	$P_x = -0.06\ N/mm^2$	
	t_x	t_y
1.152	0.028251	0.070628
0.864	-11.311	-28.276
0.576	-14.836	-37.089
0.288	-18.925	-47.313
-0.288	- 2.6371	-54.719
-0.576	2.7239	-54.14
-0.864	8.9688	-52.192
-1.152	18.528	-46.321
-1.144	14.875	-37.188
-1.728	11.191	-27.978
-2.016	- 0.4367	1.0918

x-coordinates on foundation (mm)	$P_x = -0.075\ N/mm^2$	
	t_x	t_y
0.9936	-1.0260	-2.565
0.79488	-9.7082	-24.271
0.59616	-12.853	-32.132
0.39744	-15.575	-38.937
0.19872	-18.426	-46.065
-0.19872	- 8.1476	-53.054
-0.39744	- 2.9691	-54.297
-0.59616	0.5159	-54.535
-0.79488	4.2568	-53.941
-0.9936	7.8747	-52.692
-1.19232	16.781	-49.251
-1.39104	17.449	-43.623
-1.58976	15.047	-37.617
-1.78848	12.108	-30.269
-1.98720	9.0397	-22.599
-2.18592	-0.75849	1.8961

Continued......

Table 4.5 (Continued):

x-coordinates on foundation (mm)	$P_x = -0.095\ N/mm^2$	
	t_x	t_y
0.8784	-1.0314	-2.5785
0.576	-11.819	-29.546
0.288	-15.960	-39.899
-0.288	-12.198	-52.017
-0.576	- 2.1875	-54.397
-0.864	2.5601	-53.938
-1.152	9.1389	-51.665
-1.44	18.198	-45.496
-1.728	14.430	-36.076
-2.016	10.371	-25.928
-2.268	- 0.61621	1.4505

Table 4.6 : Same Material for Cylinder and Foundation (Plane Strain). Vertical Load = 32.84 N/mm. Traction distributions on the foundation

P_y = -6 N/mm^2, μ = 0.01, ν = 0.35, E=4000.MPa, Radius=31.4 mm

x-coordinates on foundation (mm)	t_x N/mm^2	t_y N/mm^2	Applied Horizontal Traction (P_x) N/mm^2
0.7536	0.2054331E-02	-0.2054331E+00	
0.5652	0.1944955E+00	-0.1944955E+02	
0.5024	0.1905313E+00	-0.2004528E+02	
0.3768	0.7110624E-01	-0.2364964E+02	
0.2512	0.4838224E-01	-0.2575839E+02	
0.1256	-0.1167128E-02	-0.2722918E+02	-0.005
-0.1256	-0.1610054E-01	-0.2748651E+02	
-0.2512	-0.6952741E-01	-0.2628408E+02	
-0.3768	-0.1421907E+00	-0.2443732E+02	
-0.5024	-0.2134397E+00	-0.2134397E+02	
-0.6280	-0.1732978E+00	-0.1732978E+02	
-0.7536	-0.5323919E-01	-0.5323919E+01	
0.7536	0.4559113E-01	0.4559113E+01	
0.6280	0.1467963E+00	-0.1467963E+02	
0.5024	0.1450045E+00	-0.1913107E+02	
0.3768	0.5829449E-01	-0.2306485E+02	
0.2512	0.4032738E-01	-0.2546045E+02	
0.1256	-0.5558744E-02	-0.2713859E+02	-0.01
-0.1256	-0.2143195E-01	-0.2781172E+02	
-0.2512	-0.7800949E-01	-0.2684136E+02	
-0.3768	-0.1758459E+00	-0.2526812E+02	
-0.5024	-0.2271789E+00	-0.2271789E+02	
-0.6280	-0.1861081E+00	-0.1861081E+02	
-0.7536	-0.1406931E+00	-0.1406931E+02	
-0.8792	0.6769729E-01	0.6769729E+01	
0.5495	0.2396421E-01	0.2396421E+01	
0.5024	-0.4346475E-01	-0.4346475E+01	
0.3768	-0.1624831E+00	-0.1624831E+02	
0.2512	-0.1627583E+00	-0.2065658E+02	
0.1256	-0.1000938E+00	-0.2410768E+02	
-0.1256	-0.1573599E+00	-0.2742126E+02	-0.05
-0.2512	-0.2759253E+00	-0.2759253E+02	
-0.3768	-0.2728539E+00	-0.2728539E+02	
-0.5024	-0.2614240E+00	-0.2614240E+02	
-0.6280	-0.2413473E+00	-0.2413473E+02	
-0.7536	-0.2085877E+00	-0.2085877E+02	
-0.8792	-0.1674826E+00	-0.1674826E+02	
-0.10048	-0.3123619E-01	-0.3123619E+01	

Table 4.7 : Teflon Cylinder on a Steel Foundation (Plane Stre
Vertical Load = 75.3 N/mm. Traction Distribution on the Foundation

P_y = -6 N/mm^2, μ = 0.4, ν = 0.35, E_{Teflon} = 0.59GPa, Radius = 72

x-coordinates on foundation (mm)	t_x N/mm^2	t_{y_2} N/mm^2	Applied horizontal traction (P N/mm^2
0.1378058E+01	-0.3444666E+00	-0.8611666E+00	
0.1181193E+01	-0.2817621E+01	-0.7044052E+01	
0.9843272E+00	-0.3719509E+01	-0.9298772E+01	
0.7874618E+00	-0.4525787E+01	-0.1131447E+02	
0.5905963E+00	-0.5255468E+01	-0.1313867E+02	
0.3937309E+00	-0.3239295E+01	-0.1451671E+02	-0.01
0.1968654E+00	-0.1268908E+01	-0.1514729E+02	
-0.1968654E+00	0.5891160E+00	-0.1529499E+02	
-0.3937309E+00	0.1961679E+01	-0.1497321E+02	
-0.5905963E+00	0.4445024E+01	-0.1405972E+02	
-0.7874618E+00	0.4974437E+01	-0.1243609E+02	
-0.9843272E+00	0.4253855E+01	-0.1063464E+02	
-0.1181193E+01	0.3372676E+01	-0.8431690E+01	
-0.1574924E+01	-0.7954376E+00	0.1988594E+01	
0.9843272E+00	-0.1365928E+01	-0.3414821E+01	
0.7874618E+00	-0.3160373E+01	-0.7900933E+01	
0.5905963E+00	-0.4030313E+01	-0.1007578E+02	
0.3937309E+00	-0.4747751E+01	-0.1189438E+02	
0.1968654E+00	-0.5016703E+01	-0.1385351E+02	
-0.1968654E+00	-0.1055871E+01	-0.1535221E+02	-0.07
-0.3937309E+00	0.6137192E-02	-0.1534935E+02	
-0.5905963E+00	0.1116238E+01	-0.1519338E+02	
-0.7874618E+00	0.2189050E+01	-0.1482261E+02	
-0.9843272E+00	0.5044195E+01	-0.1369607E+02	
-0.1181193E+01	0.4765130E+01	-0.1191282E+02	
-0.1378058E+01	0.4029027E+01	-0.1007257E+02	
-0.1574924E+01	0.3078984E+01	-0.7697459E+01	
-0.1771789E+01	0.2094911E+01	-0.5237277E+01	
-0.1968654E+01	-0.1726547E+01	0.4316368E+01	
0.7874618E+00	-0.1666333E+01	-0.4165833E+01	
0.5905963E+00	-0.3274758E+01	-0.8186895E+01	
0.3937309E+00	-0.4111599E+01	-0.1027900E+02	
0.1968654E+00	-0.4997010E+01	-0.1249253E+02	
-0.1968654E+00	-0.3113656E+01	-0.1479780E+02	
-0.3937309E+00	-0.1080090E+01	-0.1536889E+02	
-0.5905963E+00	-0.2692308E-01	-0.1542662E+02	-0.1
-0.7874618E+00	0.1117485E+01	-0.1525008E+02	
-0.9843272E+00	0.2189340E+01	-0.1486604E+02	
-0.1181193E+01	0.4945521E+01	-0.1375445E+02	
-0.1378058E+01	0.4795838E+01	-0.1198960E+02	
-0.1574924E+01	0.4061970E+01	-0.1015493E+02	
-0.1771789E+01	0.3130260E+01	-0.7825649E+01	
-0.1968654E+01	0.2213079E+01	-0.5532697E+01	
-0.2192134E+01	-0.1559430E+01	0.3898574E+01	

CHAPTER 5

CONCLUSIONS

Based on the results and discussions presented in Chapter 4, following conclusions are made:

(i) In BEM displacements and tractions are the variables whereas in FEM displacements and forces are the variables and this fact makes BEM more suitable for contact problems.

(ii) Linear elements give better results than constant elements in case of traction fields as well as displacement fields. Moreover if the load is applied in increments and the discretization is done with linear elements, introducing previous step's displacement fields in the existing model is easier.

(iii) It is observed that solving the contact problem in natural coordinates system is preferable as it can take care of any configuration of the contact region easily. When large constact region is present, tedious matrix operations are needed if cartesian coordinates are used, but in a natural coordinate system this problem will not arise.

(iv) While all primary solutions, displacement fields and traction fields, behave well the secondary solution (moment about the centre in rolling contact) show instability.

Scope for further work:

(i) To get the secondary solution, moment, in an accurate manner alternate ways like higher precision calculation and introducing moment as a degree of freedom may be explored.

(ii) To simulate rolling wheel at all speeds boundary conditions based on 'creep' factor (if two wheels are in contact and their velocities in the contact region are V_1 and V_2 then creep, ξ, $= \dfrac{2(V_1 - V_2)}{V_1 + V_2}$) may be used.

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APPENDIX A

Boundary stresses can be calculated by a method presented by Rizzo and Shippy [17] . This method obviates the necessity to evaluate singular integrals. The method is described below.

There are seven unknowns on the boundary, viz. 3 stress components and 4 displacement gradient components ($u_{i,j}$). If seven equations are available all these unknowns can be evaluated. $p_j = \sigma_{ij} n_i$ (1)

supplies two equations while stress-strain relationships supply three equations. Two more equations can be obtained by evaluating du_i/ds numerically along the boundary and equating it to analytical expression for the same

$$\frac{du_i}{ds} = \frac{du_i}{dx_1}\frac{dx_1}{ds} + \frac{du_i}{dx_2}\frac{dx_2}{ds} \qquad (2)$$

$$= u_{i,2}\, n_1 - u_{i,1}\, n_2$$

where

n_1, n_2 - normal components in n_1 and n_2 directions

s - indicates boundary.

Along with Eqn. 2 there are seven equations for seven unknowns. Solving these linear algebraic equations stresses on the boundary can be obtained.

```
**************************************************************************
*                                                                        *
*     P R O G R A M   F O R   C O N T A C T   P R O B L E M S            *
*                                                                        *
**************************************************************************
```

B E M is used for solving the problem. Two dimensional isotropic problems are solved. Only two bodies should be in contact.Both bodies need not to be of same material. Plane stress and plane strain assumptions can be handled. Linear elements are used. This program gives the craction & displacement distributions throghout the bodies in contact ie. including the contact region.

SOME IMPORTANT VARIABLES

```
XM,YM  ----->  Coordinates of nodes on the bodies
               (as only boundary is discretised all these
                nodes will be on the boundaries)

XI,YI  ----->  Internal points of the bodies
                 (selected according to the necessity)

BV     ----->  Boundary values (traction & displacement)

KODE   ----->  Code to identify the nature of the BVs
                 ( 1  is traction & 0 is displacement)

KCONT  ----->  Variable that keeps the sequence of the
                 nodes in the contact region.
                 For eg. to the left of the vertical axis
                 of symmetry( consider a roller over a foundation)
                 if 7,8,9,10 are the nodes of
                 the upper body and 27,26,25,24 are tha
                 of the lower body and these nodes are in
                 contact in the same sequence(ie. 7 & 27
                 , 8 & 26 , ...) then first array of this
                 variable contains 24,10,25,9,26,8,27,7.
                 Similarly to the right of the axis of symmetry
                 when 11,12, 13,14 of upper body and
                 23,22,21,20 of lower body are in contact
                 in the same sequence, second array contains
                 23,11,22,12,21,13,20,14

TOTANG  ----->   Angle subtended by the line connecting
                 node and the center of the roller(upper
                 body) and the vertical axis of symmetry.
                  In the above example angles subtended
                 by nodes 10,9,8 & 7 will be the first array
                 whereas angles subtended by 11,12,13 & 14
                 will form the second array of the variable
```

```
HH,GM ------->  H & G matrices as defined in the theory.
                 [H] {u} = [G] {t}

MBV   ------->  Right side known column vector of [A] {X} = {B}
                 soln. for unknown {X} will be stored in
                 this array itself

GMOD  ------->  Shear modulus (if different materials
                 are used then this array variable will have
                 two different values. Otherwise both the values
                 will be same)

PR  ------->    Poission's ratio (for different materials
                 different values ;similar to GMOD)

AMU   ------->  Co-efficient of sliding friction
ALIM  ------->  'Zero' value . Used in determining the ends
                 of the contact region.

************************************************************************

Main program . This calls GHMATX & CNTACT subroutines

DIMENSION NODE(95),XN(96),YN(96)
DIMENSION KCONT(2,32),TOTANG(2,16),XI(20),YI(20)
COMMON/MAT/HH(193,193),GM(193,193),MBV(193,1)
COMMON/IPWT/GPP(32),WT(32)
COMMON/CNSTNT/GE1,POI1,GMOD(2),PR(2),AMU,PI
COMMON/BCS/DISPL(193),TRACT(193),BV(193),KODE(193)
COMMON/PRGVAL/NN,N1,MCH,ALIM,NOCON,IPROB,NC(2),NDUB,
1              INDB(10),INDE(10),CONTIN
COMMON/DIM/IH,IH2,IHM,IHM2,INP1
COMMON/SUBCNT/P(193),WKS(193)

REAL MBV
PI=3.14159265358

R is the radius of the wheel on the elastic foundation .

R=11.1
        IH is twice the number of nodes ; IHM is more because
        it also involves 3 secondary variables, ie. calculated
        from primary variable traction
IH=190 ; IHM=193
        INP1 is No. of nodes + 1
INP1=96
IH2= IH*IH   ; IHM2=IHM*IHM

OPEN(UNIT=23,DEVICE='DSK',FILE='CONTAT.IN')
OPEN(UNIT=24,DEVICE='DSKD',FILE='CONTAT.OUT')
OPEN(UNIT=27,DEVICE='DSK')
OPEN(UNIT=28,DEVICE='DSK')

        Almost all the data needed for calculation are read
        below. Few more read in CNTACT.

READ(23,*)CONTIN,MCH
READ(23,*)NN,N1,GMOD(1),GMOD(2),PR(1),PR(2),ALIM,NOCON,IPROB,AMU
READ(23,*) NDUB
IF(NDUB)10,10,20
READ(23,*)(INDB(I),INDE(I),I=1,NDUB)
READ(23,*) (NC(I),I=1,MCH)
```

```
IF(IL)30,30,10
READ(23,*) (XI(I),YI(I),I=1,NI)
READ(23,*) (XN(I),YN(I),I=1,NN)
READ(23,*) (NODE(I),BV(2*I-1),KODE(2*I-1),BV(2*I),KODE(2*I)
1 ,I=1,NN)
READ(23,*) ((KCONT(I,J),J=1,2*NOCON),I=1,2)
READ(23,*) ((TOTANG(K,I),I=1,NOCON),K=1,2)

For different diameter , the corresponding coordinates are
obtained from a standard set ( Rad = 50. ) by multiplying with
a factor so that most of the data in the input (ie. coordinates)
can be retained irrespective of the dia. of the roller

DO 100 I=1,NN
XN(I)=R/50.*XN(I)
YN(I)=R/50.*YN(I)
CONTINUE

GFP gaussian function points ;
WT weight at the corresponding pts.
Four gaussian pts. and Twelve gaussian points are given .
However only 12 gaussian points are used in the work

         GFP(1)=0.8611363  ;   GFP(2)=0.3399810
         GFP(4)=-0.8611363  ;   GFP(3)=-0.3399810
         WT(1)=0.3478548  ;   WT(2)=0.6521452
         WT(4)=0.3478548  ;   WT(3)=0.6521452

GFP(1)=0.1252334085        ;WT(1)=0.2491470458
GFP(2)=-GFP(1)             ;WT(2)=WT(1)
GFP(3)=0.3678314989982     ;WT(3)=0.23349253654
GFP(4)=-GFP(3)             ;WT(4)=WT(3)
GFP(5)=0.5873179543        ;WT(5)=0.203167426723
GFP(6)=-GFP(5)             ;WT(6)=WT(5)
GFP(7)=0.7699026742        ;WT(7)=0.160078328543
GFP(8)=-GFP(7)             ;WT(8)=WT(7)
GFP(9)=0.9041172563 7      ;WT(9)=0.106939325995
GFP(10)=-GFP(9)            ;WT(10)=WT(9)
GFP(11)=0.9815606342467    ;WT(11)=0.0471753363865
GFP(12)=-GFP(11)                 ;WT(12)=WT(11)

         PAUSE 'ENTERING GHMATX'
         System matrices [H] & [G] are calculated
CALL GHMATX(XN,YN)

[H] & [G] are stored in a file for reusing as long as the
coordinates of the nodes do not change. If coordintes change
then fresh [H] & [G] will be written in the same file

REWIND 27
WRITE(27) HM
WRITE(27) GM


NBC=2*NN
NBCFIX=NBC
         PAUSE 'ENTERING CNTACT'
         Almost all the work is done by CNTACT
CALL CNTACT(TOTANG,KCONT,XN,YN,XI,YI,CONLEN,PEAK)

STOP;END
```

```
*****************************************************************

This routine introduces the contact conditions and rearranges
the matrices in such a way that finally [A] {X} = [B] form is
obtained and this is solved by LU decomposition method. This
routine also maintains the iterations till needed.

SUBROUTINE CNTACT(TOTANG,KCONT,XN,YN,XI,YI,CONLEN,PEAK)

DIMENSION XN(INP1),YN(INP1),KCONT(2,32),NO(2),CONTAT(2,16)
DIMENSION KODSLP(2,20),NDSLIP(2,16),COMP(2,16)
DIMENSION TOTANG(2,16),DEG1(2),DEG2(2),XI(20),YI(20)
DIMENSION LOCAT1(50),LOCAT2(50),RD(2,16)

COMMON/MAT/HM(193,193),GM(193,193),MBV(193,1)
COMMON/IPWT/GFP(32),WT(32)
COMMON/CNSTNT/GE1,POI1,GMOD(2),PR(2),AMU,PI
COMMON/BCS/DISPL(193),TRACT(193),BV(193),KODE(193)
COMMON/PRGVAL/NN,NI,MCH,ALIM,NOCON,IPROB,NC(2),NDUB,
1           INDB(10),INDE(10),CONTIN
COMMON/TRANS/NODUP(2),NODOWN(2),ALOCAT(2),HANG(2),SEARCH(2)
COMMON/DIM/IH,IH2,IHM,IHM2,INP1
COMMON/SUBCNT/P(193),WKS(193)

READ MBV

GE=GMOD(1)
R=31.1
NBCFIX=2*NN
NBC=NBCFIX

        The following input helps starting the program from an
        intermediate stage .

READ(23,*)ITER,NO(1),NO(2),NODLEF,NODRIT,IRFIX

DO 820 I=1,NOCON
CONTAT(1,I)=TOTANG(1,I)-TOTANG(1,I-1)
CONTAT(2,I)=TOTANG(2,I)-TOTANG(2,I-1)
CONTINUE

        Beginning of iterative loop

KINC=0

REWIND 27
READ(27) HM
READ(27) GM

        NO(1) no. of nodes in contact on the left side of
        vertical axis of symmetry and NO(2) similarly on the
        right side

TYPE *,ITER,NO(1),NO(2)

        CONLEN is total contact length.

CONLEN=ABS(XN(KCONT(1,2*NO(1)-1)))+ABS(XN(KCONT(2,2*NO(2)-1)))
NTOT = 2 *NN
MOMENT increases the size of matrices [HM] & [GM] from 2N x 2N
        to (2N + 3) x (2N + 3)

CALL MOMENT(KINC,LOCAT1,LOCAT2,KCONT,XN,YN,NTOT,R,NO)
```

```
DO 650 I=1,NTOT
MBV(I,1)=0.

NBC3=NBCFIX

    IF(LFIX.EQ.0)NO(1)=0
     IF(IRFIX.EQ.0)NO(2)=0
```

The following loop (65 DO ... CONTINUE)introduces the contact conditions .

```
DO 65    NUM=1,ITER
DO 65 K=1,2

    IF(K.EQ.1.AND.LFIX.EQ.0)GO TO 68
     IF(K.EQ.2)GO TO 78
      IF(NUM.GT.NO(1))GO TO 65
    GO TO 63

    NO(1)=NO(1)+1
    GO TO 63

     IF(IRFIX.EQ.0)GO TO 69
      IF(NUM.GT.NO(2))GO TO 65
     GO TO 63

     NO(2)=NO(2)+1

CONTINUE

     IT=NUM-1
     FRIC=AMU
```

NDSLIP has the numbers of the nodes that are slipping

```
IF(NDSLIP(K,NUM).EQ.KCONT(K,2*(NUM-1)+2))GO TO 575
//SDIS=(XN(KCONT(K,2*NUM-1))+CONLEN)*ZETA//
//IF(ITRADD.EQ.0) SDIS=0.//

          DO 650 I=1,NTOT
           GM(I,2*KCONT(K,2*NUM)-1)=
1          GM(I,2*KCONT(K,2*NUM)-1)-GM(I,2*KCONT(K,2*(NUM-1)+1)-1)
           HM(I,2*KCONT(K,2*NUM)-1)=
1          HM(I,2*KCONT(K,2*NUM)-1)+HM(I,2*KCONT(K,2*(NUM-1)+1)-1)
        IF(N1)101,101,102
       MBV(I,1)=MBV(I,1)+HM(I,2*KCONT(K,2*(NUM-1)+1)-1)*DELSLP
       GO TO 103
       MBV(I,1)=MBV(I,1)+HM(I,2*KCONT(K,2*(NUM-1)+1)-1)*DELSLP
       CONTINUE
           GM(I,2*KCONT(K,2*NUM))=
1           GM(I,2*KCONT(K,2*NUM))-GM(I,2*KCONT(K,2*(NUM-1)+1))
            MBV(I,1)= MBV(I,1) - HM(I,2*KCONT(K,2*(NUM-1)+1))*
1             (YN(KCONT(K,2*(NUM-1)+2))-YN(KCONT(K,2*(NUM-1)+1)))
           HM(I,2*KCONT(K,2*NUM))=
1           HM(I,2*KCONT(K,2*NUM))+HM(I,2*KCONT(K,2*(NUM-1)+1))
         CONTINUE

     NBCAT1(KINC+1)=2*KCONT(K,2*(NUM-1)+1)-1
     NBCAT2(KINC+1)=2*KCONT(K,2*NUM)-1

     NBCAT1(KINC+2)=2*KCONT(K,2*(NUM-1)+1)
```

```
      LOCAT2(KINC+2)=2*KCONT(K,2*NUM)

          DO 655 I=1,NTOT
           HM(I,2*KCONT(K,2*(NUM-1)+1)-1)=
     1      - GM(I,2*KCONT(K,2*NUM)-1)*GE
          HM(I,2*KCONT(K,2*(NUM-1)+1))=
     1      - GM(I,2*KCONT(K,2*NUM))*GE
         CONTINUE

       KINC=KINC+2
     GO TO 665

     IF(IADD.GT.0.AND.NUM.EQ.NO(K))GO TO 411
     IF(KADD.GT.0.AND.NUM.EQ.NO(K))GO TO 411
     GO TO 397

     CONTINUE
         FRIC=COMP(K,NO(K)-1)*AMU
     WRITE(5,*) FRIC
     GO TO 390
     CONTINUE

     IF(TRACT(2*KCONT(K,2*(NUM-1)+2)-1)*TRACT(2*KCONT
    1(K,2*(NUM-1)+2)).GT.0.)FRIC=-AMU

     WRITE(5,*) FRIC
            DO 660 I=1,NTOT
            JK=NUM
            IF(NUM.EQ.NO(K))JK=NO(K)-1
             GM(I,2*KCONT(K,2*NUM)-1)=
     1       GM(I,2*KCONT(K,2*NUM)-1)+GM(I,2*KCONT(K,2*NUM))/
     2         (-AMU*COMP(K,JK))
             GM(I,2*KCONT(K,2*NUM)-1)=
     1       GM(I,2*KCONT(K,2*NUM)-1)-GM(I,2*KCONT(K,2*(NUM-1)+1)-1)
             GM(I,2*KCONT(K,2*NUM)-1)=
     1        GM(I,2*KCONT(K,2*NUM)-1)+GM(I,2*KCONT(K,2*(NUM-1)+1))/
     2         (AMU*COMP(K,JK))
              MBV(I,1)= MBV(I,1) - HM(I,2*KCONT(K,2*(NUM-1)+1))*
     1          (YN(KCONT(K,2*(NUM-1)+2))-YN(KCONT(K,2*(NUM-1)+1)))
             HM(I,2*KCONT(K,2*NUM))=
     1        HM(I,2*KCONT(K,2*NUM))+HM(I,2*KCONT(K,2*(NUM-1)+1))
          HM(I,2*KCONT(K,2*(NUM-1)+1))= -GM(I,2*KCONT(K,2*NUM)-1)*
     1      GE
            CONTINUE

            LOCAT1(KINC+1)=2*KCONT(K,2*(NUM-1)+1)
            LOCAT2(KINC+1)=2*KCONT(K,2*NUM)-1

     KINC=KINC+1

     CONTINUE
     TYPE *, KINC


     CONTINUE

     WRITE(5,*)(LOCAT1(I),I=1,KINC)
     WRITE(5,*)(LOCAT2(I),I=1,KINC)

       Specified boundary conditions are implimented below

     DO 35 J=1,NBCFIX
```

```
    IF (KODE(J).NE.0) GO TO 35

                  DO 40 I=1,NBCFIX
                     VALUE=HM(I,J)
                     HM(I,J)=-GM(I,J)*GE
                     GM(I,J)=-VALUE
                  CONTINUE

   CONTINUE
```

Right side of [A] {X} = { B } is found out

```
   CALL MULTI(IH,IH2,IHM,IHM2)

   NBC3 = NTOT
   TYPE *,NBC3,NBCFIX
   PAUSE4

   CALL MATIN(HM,NBC3,MBV,1,DET)

   CALL F04ARF(HM,IHM,MBV,NBC3,SOL,WKS,IFAIL)
   DO 209 I=1,NBC3
   MBV(I,1)=SOL(I,1)
   CONTINUE
```

Solution of [A] { X } = { B } by LU decomposition

```
   CALL F01BTF(NBC3,HM,IHM,P,DP,IFAIL)
           TYPE *,IFAIL,DP

   CALL F04AYF(NBC3,1,HM,IHM,P,MBV,IHM,IFAIL)
           TYPE *,IFAIL
```

Output below . Using variables LOCAT1 , LOCAT2 & KODE values in MBV ({X}) are properly related to the corresponding nodes.

```
   DO 110 K=1,2*NN

      DO 115 J=1,KINC
      IF(K.EQ.LOCAT1(J))GO TO 670

     IF(KODE(K)) 125,125,130

          DISPL(K)=BV(K)
          TRACT(K)=MBV(K,1)*GE
          GO TO 120

      DISPL(K)=MBV(K,1)
      TRACT(K)=BV(K)
      CONTINUE
      GO TO 110

    IF(LOCAT2(J) .EQ.  LOCAT2(J)/2*2) GO TO 671
           TRACT(LOCAT2(J)) = MBV(LOCAT1(J),1) * GE
           GO TO 672
              TRACT(LOCAT2(J)) = MBV(LOCAT1(J),1) * GE
```

```
        CONTINUE

CONTINUE
```

Retrieving the tractions and displacements of nodes other than those found out from direct solution with the help of contact conditions.

```
    DO 675 K=1,2
      DO 675 J=1,NO(K)
//SDIS=(XN(KCONT(K;2*NUM-1))+CONLEN)*ZETA//
        JK=J
        IF(J.EQ.NO(K))JK=NO(K)-1

        TRACT(2*KCONT(K,2*(J-1)+1)-1)= -TRACT(2*KCONT(K,2*J)-1)
        TRACT(2*KCONT(K,2*(J-1)+1))= -TRACT(2*KCONT(K,2*J))
        DISPL(2*KCONT(K,2*(J-1)+1))= DISPL(2*KCONT(K,2*J))+
1            (YN(KCONT(K,2*(J-1)+2))-YN(KCONT(K,2*(J-1)+1)))
     IF(NOSLIP(K,J).EQ.KCONT(K,2*J)) GO TO 680
        GO TO 685
        TRACT(2*KCONT(K,2*J))=(TRACT(2*KCONT(K,2*J)-1)/
1                              (-AMU*COMP(K,JK)))
        TRACT(2*KCONT(K,2*(J-1)+1))= -TRACT(2*KCONT(K,2*J))
        GO TO 675
        DISPL(2*KCONT(K,2*(J-1)+1)-1)=DISPL(2*KCONT(K,2*J)-1)
     IF(NN)104,104,105
         DISPL(2*KCONT(K,2*(J-1)+1)-1) =
1                  DISPL(2*KCONT(K,2*(J-1)+1)-1)-DELSLP
        GO TO 106
         DISPL(2*KCONT(K,2*(J-1)+1)-1) =
1                  DISPL(2*KCONT(K,2*(J-1)+1)-1)-DELSLP
       CONTINUE

CONTINUE
```

Secondary variables viz. Force-x , Moment due to Force-x, Moment due to Force-y are also directly obtained by introducing appropriate equations in the matrix system.
Alternate way is to calculate separately from the results.

```
    AMX = MBV(NBC3-2,1)
    FY  = MBV(NBC3-1,1)
    AMY = MBV(NBC3,1)

WRITE(24,707) AMX,AMY,FY
FORMAT(/,10X,'MOMENT DUE TO X-FORCE  =  ',E20.8,//,
1          10X,'MOMENT  ,,  ,, Y-FORCE  =  ',E20.8,//,
2          10X,'Y-FORCE IN CONTACT AREA=  ',E20.8,///)
```

Following binary file stores important input & output values for future use in other programs.

```
          REWIND (28)
           WRITE(28) XN,YN,NC,NI,MCH,XI,YI,TRACT,DISPL

IF (NN.EQ.1) GO TO 108
 GO TO 198
```

COMP stores the sign of AMU in each slipping node.
Criteria is Tx + AMU * Ty = 0

```
DO 398 K=1,2
    DO 398 J=1,NO(K)
```

```
          I=2*(J-1)+2
    IF((TRACT(2*KCONT(K,I)-1)*TRACT(2*KCONT(K,I))).GT.0.)GO TO 399
          COMP(K,J)=1.
      GO TO 398

?
          COMP(K,J)=-1.

    CONTINUE

    GO TO 109
    DO 107 K=1,2
       DO 107 J=1,NO(K)
         IF(K.EQ.1)COMP(K,J)= 1.
         IF(K.EQ.2)COMP(K,J)=-1.
    CONTINUE

    WRITE(5,530)((I,DISPL(2*I-1),DISPL(2*I),TRACT(2*I-1),TRACT(2*I)
 1  ),I=20,40)

    WRITE(5,777)
    FORMAT(5X,' TYPE IN ZERO VALUE ')

    ACCEPT *, ALIM
    IF(INDIC.EQ.1)GO TO 544

    IF(TRACT(2*KCONT(1,2*(NO(1)-1)+1)).GT.0.
 1  .OR.ABS(TRACT(2*KCONT(1,2*(NO(1)-1)+1))).LT.ALIM)GO TO 541

          If LFIX = 0 left side contact region (w.r.t vertical
      axis of symmetry) can grow . If LFIX = 1 no more growth
      allowed . Similarly for right side IRFIX is used for
      controlling growth.

       IF(IPRIL)932,932,933
          LFIX=0
          GO TO 542
             NO(1)=NO(1)+1
             NODLEF=NODLEF+1
          GO TO 542

    LFIX=1 ; IPRIL=1

    IF(TRACT(2*KCONT(2,2*(NO(2)-1)+1)).GT.0.
 1  .OR.ABS(TRACT(2*KCONT(2,2*(NO(2)-1)+1))).LT.ALIM)GO TO 543

        IF(IPRIR)930,930,931
          IRFIX=0
          GO TO 544
             NO(2)=NO(2)+1
             NODRIT=NODRIT+1
          GO TO 544

    IRFIX=1 ; IPRIR=1


    WRITE(24,525)
    FORMAT('NODE NO.',5X,'DISPLACEMENT X',5X,'DISPLACEMENT Y',5X,
 1  'TRACTION X',5X,'TRACTION Y',//)
    WRITE(24,530)((I,DISPL(2*I-1),DISPL(2*I),TRACT(2*I-1),TRACT(2*I)
 1  ),I=1,NN)
    FORMAT(3X,I5,5X,E12.5,7X,E12.5,7X,E12.5,3X,E12.5)
```

```
      GO TO 400

            CONTINUE
          DO 114 K=1,2
            DO 116 J=1,NO(K)
        RD(K,J)=DISPL(2*KCONT(K,2*J)-1)-DISPL(2*KCONT(K,2*(J-1)+1)-1)
        WRITE(24,117)(RD(K,JKL),JKL=1,NO(K))

      PAUSE 'ENTERING SLIP'

      CALL SLIP(KCONT,NDSLIP,KODSLP,IADD,
     1      JADD,KADD,ITRADD,ITER,NO,DELSLP,MSLIP,IK1,IK2)
C     For static contact problems activate the following 2 statements

C     delslp = 0.
C     go to 203

C     MSLIP helps in controlling the a subloop of iteration to
C     introduce the 'relative displacement' in no-slip region also
C     to simulate rolling contact conditions.

        MSLIP=1
            IF((IK1.EQ.0) .AND. (IK2.GT.0))MSLIP=0
        RELDIF=REL-DELSLP
         RRDIF=RR-RELDIF
        WRITE(5,112) RELDIF,RRDIF
       FORMAT(5X,'RELDIF = ',E16.5,/,5X,'RRDIF = ',E16.5)
        IF (ABS(RRDIF).GE.0.00001)GO TO 5
            MSLIP=0 ; REL=0. ; RR=0.
      CALL SLIP(KCONT,NDSLIP,KODSLP,IADD,
     1      JADD,KADD,ITRADD,ITER,NO,DELSLP,MSLIP,IK1,IK2)

        GO TO 203
        CONTINUE
            REL=DELSLP
            RR=RELDIF
        WRITE(5,*)(((NDSLIP(K,I),KODSLP(K,I)),I=1,ITER),K=1,2)

      FORMAT(/,5X,8E14.4)
         TYPE *, IADD
         TYPE *, MSLIP
      PAUSE 'AFTER ASSESSING SLIP'
      IF(IADD.EQ.0)GO TO 860

        NH=0
      GO TO 155


      ILE=KCONT(1,2*NO(1)-1); ILE2=KCONT(1,2*(NO(1)-1)-1)
      IRE=KCONT(2,2*NO(2)-1); IRE2=KCONT(2,2*(NO(2)-1)-1)
        WRITE(5,*)ITER,TRACT(2*ILE),TRACT(2*IRE),TRACT(2*ILE2)
        WRITE(5,*)TRACT(2*ILE-1),TRACT(2*IRE-1)
      NODLFT=NO(1)
      NODRAT=NO(2)
      GO TO 604

        DO 601 K=1,2
              GO TO (602,603),K

            IF(TRACT(2*ILE).GT.TRACT(2*ILE2))NO(1)=NO(1)-1
```

```
      GO TO 601

      IF(TRACT(2*IRE).GT.TRACT(2*IRE2))NO(2)=NO(2)-1

   CONTINUE

CONTINUE

IF(NODLEF.NE.NO(1).OR.NODRIT.NE.NO(2))GO TO 155

IF((TRACT(2*ILE)).GE.0..AND.TRACT(2*IRE).GE.0.)GO TO 153
IF(ABS(TRACT(2*ILE)).LE.ALIM.AND.TRACT(2*IRE).GE.0.)GO TO 153
IF((TRACT(2*ILE)).GE.0..AND.ABS(TRACT(2*IRE)).LE.ALIM)GO TO 153
IF(ABS(TRACT(2*ILE)).LE.ALIM.AND.
1         ABS(TRACT(2*IRE)).LE.ALIM)GO TO 153

          IF(INDIC.EQ.1)GO TO 153

             ITER=ITER+1
               TYPE *,ITER
          IF(ITER.EQ.NOCON+1) GO TO 888

             IF(LFIX.EQ.0)NO(1)=NO(1)+1
              IF(IRFIX.EQ.0)NO(2)=NO(2)+1
                  TYPE*, LFIX,IRFIX
          GO TO 155


          IF(ABS(TRACT(2*ILE)).LT.ALIM.AND.
1            ABS(TRACT(2*IRE)).LT.ALIM)GO TO 540
           IF(ITRADD.EQ.0)GO TO 540

CALL ZELUC(NO(1),NO(2),TOTANG,CONTAT,IND,INDIC,XN,YN,R,ITRADD,
1           DEG1,DEG2,KCONT)

CALL GNMATX(XN,YN)
GO TO 155
CONTINUE

   NN=1

DO 790 K=1,2
   DO 790 J=1,NO(K)
          IF(K.EQ.1)COMP(K,J)= 1.
          IF(K.EQ.2)COMP(K,J)= 1.
CONTINUE

GO TO 155
CONTINUE
PERR=TRACT(2*KCONT(1,2))
WRITE(24,649)  CONGEN
FORMAT(10X,'CONTACT LENGTH =  ',E20.9,//)

WRITE(24,160)
FORMAT('NODE NO.',5X,'DISPLACEMENT X',5X,'DISPLACEMENT Y',5X,
1  'TRACTION X',5X,'TRACTION Y',//)

WRITE(24,165)((I,DISPL(2*I-1),DISPL(2*I),TRACT(2*I-1),TRACT(2*I)
1  ),I=1,[illegible])
FORMAT(5X,I5,8X,E12.5,7X,E12.5,7X,E12.5,3X,E12.5)

GO TO 659
```

```
WRITE(24,887)
FORMAT(10X,'CONTACT ELEMENTS NOT ENOUGH ???????')
RETURN;END
```

```
************************************************************************
*                                                                      *
*                                                                      *
*   This program gets the input values from CONTACT program  are      *
*  used in this program to calculate stresses at internal points      *
*  as well as tnstresses at the boundary.                             *
*                                                                      *
*                                                                      *
************************************************************************

*       MAIN program . Some of the values have to be hand supplied.

*       IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION XE(97),YE(97),XI(20),YI(20),TRACT(193),DISPL(193),
     1            NC(2),GFP(32),WT(32),INDB(10),INDE(10)

        COMMON/A/GFP,WT,PI
        COMMON/B/GE,POI
        PI=3.1415927

        OPEN(UNIT=23,DEVICE='DSK',FILE='STRESS.IN')

*       NE  - No. of nodes
*       GE      - Shear modulus of the roller (GE of foundation should
*                 should be given if stresses in the foundation have to
*                 be found out.Similarly for POI also)
*       POI     - Poisson's  ratio of the roller .
*       PEAK    - Peak value of traction-y ; used to calculate stresses
*                 analytically when loading conditions are symmetrical.
*                 (Soln. by Smith  & Liu are used)
*       CONLEN - Half contact length ; supplied to EXACT subroutine.
*       NDUB    - No. of double nodes.
*       INDB    - Double node beginning
*       INDE    - Double node ending

        CONLEN=4. ; PEAK=93.    ;  NE=96  ; GE=1481.48 ; POI=0.35

        OPEN(UNIT=28,DEVICE='DSK')
        REWIND 28
        READ(28)XE,YE,NC,NI,MCH,XI,YI,TRACT,DISPL

*       Values of NDUB, INDB, & INDE are read from STRESS.IN data file.

        READ(23,*)NDUB
        READ(23,*)(INDB(I),INDE(I),I=1,NDUB)

        NI = 20

*       If we wish to give internal points other than those from
*       main file, following statments have to be used. Otherwise
*       they deactivated.

        DO 99 I=1,NI
        READ(23,*)XI(I),YI(I)
99      WRITE(24,*) XI(I),YI(I)

        DO 34 I=1,NE
34      WRITE(24,*)(XE(I),YE(I))
        WRITE(24,*)NI,MCH
```

```
          DO 17 I=1,NE
17        WRITE(24,*) I,DISPL(2*I-1),DISPL(2*I),TRACT(2*I-1),TRACT(2*I)
          PAUSE

*         STRSLN is used to calculate stresses at internal points
*         when linear elements are used for discretization.

          CALL STRSLN(NDUB,INDB,INDE,XE,YE,XI,YI,NI,TRACT,DISPL,NC)

*         INTU is for internal displacements!
                                                    For constant elements.
*         INTRNL  is for internal stresses. !

*         CALL INTU(NI,NE,NC,XE,YE,XI,YI,POI,GE,TRACT,DISPL,MCH)
*         CALL INTRNL(NI,XI,YI,XE,YE,TRACT,DISPL,NC,MCH,NE)

*         For both type elements analytical results can be obtained
*         using EXACT subroutine.(limitations are explained in thesis book)

*         CALL EXACT(CONLEN,PEAK,XI,YI,NI)
          STOP;END

*         *************************************************************

*         Routine ELSTRS is used to calculate the contributions of
*         each element in the stress calculation.(For Linear elements)

          SUBROUTINE ELSTRS(N1,N2,XN,YN,XE1,YE1,XE2,YE2,
          1 D111,D211,D112,D212,D122,D222,S111,S211,S112,S212,S122,S222,
          2 TRACT,DISPL)
*         IMPLICIT REAL*8(A-H,O-Z)
          DIMENSION XW(32),YW(32),ANORM(2),FI(2)
          DIMENSION GFP(32),WT(32)
          DIMENSION DISPL(193),TRACT(193),BV(193),KODE(193)
          COMMON/A/GFP,WT,PI
          COMMON/B/GE,POI
          ELENGT=SQRT((XE2-XE1)**2+(YE2-YE1)**2)
          ANORM(1)=(YE2-YE1)/ELENGT
          ANORM(2)=(XE1-XE2)/ELENGT
          D111=0.;D211=0.;D112=0.;D212=0.;D122=0.;D222=0.
          S111=0.;S211=0.;S112=0.;S212=0.;S122=0.;S222=0.
                    CALL GAUSS(XN,YN,XE1,YE1,XE2,YE2,NGP,GFP,WT)
          DO 20 K=1,NGP

                    FI(1)=-0.5*(GFP(K)-1.)
                    FI(2)= 0.5*(GFP(K)+1.)

               TX=FI(1)*TRACT(2*N1-1)+FI(2)*TRACT(2*N2-1)
                 TY=FI(1)*TRACT(2*N1)+FI(2)*TRACT(2*N2)
               UX=FI(1)*DISPL(2*N1-1)+FI(2)*DISPL(2*N2-1)
                 Uy=FI(1)*DISPL(2*N1)+FI(2)*DISPL(2*N2)

          XW(K)=(XE2+XE1)/2.+GFP(K)*(XE2-XE1)/2.0
          YW(K)=(YE2+YE1)/2.+GFP(K)*(YE2-YE1)/2.0
          RDIS=SQRT((XN-XW(K))**2+(YN-YW(K))**2)
          R1=(XW(K)-XN)/RDIS
          R2=(YW(K)-YN)/RDIS
C         PAUSE
          DRDN=R1*ANORM(1)+R2*ANORM(2)
          F=2.*GE/(RDIS**2*3.*PI*(1.-POI))
          C=1./(4.*PI*(1.-POI)*RDIS)
          E=1.-2.*POI
          D111=D111+C*(E*R1+2.*R1**3)*WT(K)*ELENGT/2.*Tx
```

```
      D211=D211+C*((-E)*R2+2.*R1**2*R2)*WT(K)*ELENGT/2.*Ty
      D112=D112+C*(E*R2+2.*R1**2*R2)*WT(K)*ELENGT/2.*Tx
      D212=D212+C*(E*R1+2.*R2**2*R1)*WT(K)*ELENGT/2.*Ty
      D122=D122+C*((-E)*R1+2.*R2**2*R1)*WT(K)*ELENGT/2.*Tx
      D222=D222+C*(E*R2+2.*R2**3)*WT(K)*ELENGT/2.*Ty
      S111=S111+F*(2.*DRDN*(E*R1+POI*2.*R1-4.*R1**3)+2.*POI*(2.*
     1 ANORM(1)*R1**2)+E*(2.*ANORM(1)*R1**2+2.*ANORM(1))-
     2 (1.-4.*POI)*ANORM(1))*WT(K)*ELENGT/2.*Ux
      S211=S211+F*(2.*DRDN*(E*R2-4.*R1**2*R2)+4.*POI*ANORM(1)*R1*R2
     1 +2.*E*ANORM(2)*R1**2-(1.-4.*POI)*ANORM(2))*WT(K)*ELENGT/2.*Uy
      S112=S112+F*(2.*DRDN*(POI*R2-4.*R1**2*R2)+2.*POI*(ANORM(1)*
     1 R1*R2+ANORM(2)*R1**2)+E*(2.*ANORM(1)*R1*R2+ANORM(2)))*
     2 WT(K)*ELENGT/2.*Ux
      S212=S212+F*(2.*DRDN*(POI*R1-4.*R1*R2**2)+2.*POI*(ANORM(1)*
     1 R2**2+ANORM(2)*R1*R2)+E*(2.*ANORM(2)*R1*R2+ANORM(1)))*
     2 WT(K)*ELENGT/2.*Uy
      S122=S122+F*(2.*DRDN*(E*R1-4.*R1*R2**2)+4.*POI*ANORM(2)*R1*R2+
     1 2.*E*ANORM(1)*R2**2-(1.-4.*POI)*ANORM(1))*WT(K)*ELENGT/2.*Ux
      S222=S222+F*(2.*DRDN*(E*R2+POI*2.*R2-4.*R2**3)+2.*POI*(2.*
     1 ANORM(2)*R2**2)+E*(2.*ANORM(2)*R2**2+2.*ANORM(2))-
     2 (1.-4.*POI)*ANORM(2))*WT(K)*ELENGT/2.*Uy
20    CONTINUE
      RETURN;END
*************************************************************************
*
*     Contributions from ELSTRS are asssembled in this routine
*
      SUBROUTINE STRSLD(NDUB,INDB,INDE,XN,YN,XI,YI,NI,TRACT,DISPL,NC)
*     IMPLICIT REAL*8(A-H,O-Z)
*
      DIMENSION XN(97),YN(97),XI(20),YI(20)

      DIMENSION GFP(32),WT(32),INDB(10),INDE(10),NC(2)
      DIMENSION DISPL(193),TRACT(193),BV(193),KODE(193)
      COMMON/A/GFP,WT,PI
      COMMON/B/GE,POI

      IBODY=1

      DO 20 I=1,NI
          ZMA11=0. ; ZMA12=0. ;ZMA22=0.
        DO 15 J=1,NC(1)

          DO 10 K=1,NDUB
10           IF(J.EQ.INDB(K))GO TO 15

             NBEG=J      ;     NEND=J+1
             NPRE=NC(IBODY-1)
             IF(IBODY.EQ.1) NPRE=0
             IF(NEND.GT.NC(IBODY))NEND=NEND-NC(IBODY)+NPRE

      CALL ELSTRS(NBEG,NEND,XI(I),YI(I),XN(NBEG),YN(NBEG),
     1           XN(NEND),YN(NEND),D111,D211,D112,D212,D122,D222,
     2           S111,S211,S112,S212,S122,S222,TRACT,DISPL)

             ZMA11=ZMA11+D111+D211-S111-S211
             ZMA12=ZMA12+D112+D212-S112-S212
             ZMA22=ZMA22+D122+D222-S122-S222

15    CONTINUE

      WRITE(29,25) I,XI(I),YI(I),ZMA11,ZMA12,ZMA22
25    FORMAT(5X,I4,3X,5E14.5)
```

```
20        CONTINUE

*         Stresses at the boundary are calculated below

          DO 49 I=1,NC(1)
             ND=I
          CALL BNDRY(NDUB,INDB,INDE,ND,XN,YN,TRACT,DISPL,NC)
*         WRITE(24,400) ND,BOUN11,BOUN12,BOUN22
400       FORMAT(10X,I4,5X,3E17.8)
49        CONTINUE

          RETURN;END
*         ******************************************************************

*         BNDRY routine is used for calculating stresses at the boundary.
*         Called routine is SIGBDY

          SUBROUTINE BNDRY(NDUB,INDB,INDE,ND,X,Y,T,U,NC)

*         IMPLICIT REAL*8(A-H,O-Z)

          DIMENSION X(97),Y(97),T(193),U(193),NC(2),GFP(32),WT(32)
          DIMENSION INDB(10),INDE(10)
          COMMON/A/GFP,W1,PI
          COMMON/B/GE,POI

*         WRITE(5,790)
790       FORMAT(10X,'PROGRAM IN BOUNDY SUBROUTINE')

          DO 30 I=1,NDUB
30        IF(INDE(I).EQ.ND) GO TO 20

**2       IF NUMBER OF SURFACES CHANGE FOLLOWING STATEMENT
**2       SHOULD BE CHANGED ACCORDINGLY

          NCON1=0;NCON2=NC(1)
          DO 20 I=NCON1+1,NCON2
                    IF(I.NE.ND)GO TO 20
          DO 40 J=1,NDUB
40        IF(INDB(J).EQ.ND)GO TO 50
          GO TO 60
50        K=1
          IMIN1=I-1
          IPLUS1=I+2
          IPLUS2=I+3
          GO TO 70

60        K=1
          IMIN1=I-1
          IPLUS1=I+1
          IPLUS2=I+2
70        IF(I.EQ.NCON1+1) IMIN1=NCON2
          IF(I.EQ.NCON2-1) IPLUS2=NCON1+1
          IF(I.NE.NCON2) GO TO 10
          IPLUS1=NCON1+1
          IPLUS2=NCON1+2
10        I1=2*I-1;I2=2*I
          I01=2*IMIN1-1;I02=2*IMIN1
          I11=2*IPLUS1-1;I12=2*IPLUS1
```

```
      CALL SIGBDY(X(IMIN1),X(K),X(IPLUS1),X(IPLUS2),Y(IMIN1),Y(K),
     1 Y(IPLUS1),Y(IPLUS2),U(I01),U(I1),U(I11),T(I1),U(I02),U(I2),
     2 U(I12),T(I2),BOUN11,BOUN12,BOUN22)

      WRITE(24,*) I,BOUN11,BOUN12,BOUN22

20    CONTINUE
      RETURN
      END
****************************************************************************
*
*     Finds the stresses at each node and passes them to BNDRY
*
      SUBROUTINE SIGBDY(XMIN1,X0,X1,X2,YMIN1,Y0,Y1,Y2,U10,U11,U12,T1,
     1 U20,U21,U22,T2,ZMA11,ZMA12,ZMA22)
*
      IMPLICIT REAL*8(A-H,O-Z)
*
      DIMENSION COF(7,7),WKS(7),CV(7)
      COMMON/S/GE,POI

      ELEN0=SQRT((X0-XMIN1)**2+(Y0-YMIN1)**2)
      ELEN1=SQRT((X1-X0)**2+(Y1-Y0)**2)
      ELEN2=SQRT((X2-X1)**2+(Y2-Y1)**2)

      DO 20 I=1,7
      CV(I)=0.
      DO 20 J=1,7
      COF(I,J)=0.
20    CONTINUE

      CV(1)=((U11-U10)/(ELEN0)+(U12-U11)/(ELEN1))*0.5
      CV(2)=((U21-U20)/(ELEN0)+(U22-U21)/(ELEN1))*0.5
      CV(3)=T1 ; CV(4)=T2
      XN=((Y1-Y0)/ELEN1+(Y0-YMIN1)/ELEN0)*0.5
      YN=((X0-X1)/ELEN1+(XMIN1-X0)/ELEN0)*0.5
      COF(1,1)=-YN;COF(1,2)=XN
      COF(2,3)=-YN;COF(2,4)=XN
      COF(3,5)=XN;COF(3,6)=YN
      COF(4,6)=XN;COF(4,7)=YN
      S11=1./(2.*(1.+POI)*GE);S12=-POI*S11
      COF(5,1)=-1.;COF(5,5)=S11;COF(5,7)=S12
      COF(6,4)=-1.;COF(6,5)=S12;COF(6,7)=S11
      COF(7,2)=-1.;COF(7,3)=-1.;COF(7,6)=1./GE

      CALL F01BTF(7,COF,7,WKS,DP,0)
      CALL F04AYF(7,1,COF,7,WKS,CV,7,0)

            ZMA11=CV(5)
            ZMA12=CV(6)
            ZMA22=CV(7)

      RETURN
      END
****************************************************************************
*
*     GAUSS can select preferred no. of points of Gaussian quadrature.
*     Proximity of the node and the internal point decides the no. of
*     Gaussian points.
*
      SUBROUTINE GAUSS(XP,YP,X1,Y1,X2,Y2,NGP,GFP,WT)
*     IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION GFP(32),WT(32)
```

```
*         EXTERNAL D01BAZ
.
          RREL=SQRT(((X1+X2)/2.-XP)**2+((Y1+Y2)/2.-YP)**2)
          ELENTH=SQRT((X2-X1)**2+(Y2-Y1)**2)
          FACTOR=RREL/ELENTH
          NGP=32
          IF(FACTOR.GT..2) NGP=24
          IF(FACTOR.GT..4) NGP=16
          IF(FACTOR.GT..6) NGP=12
          IF(FACTOR.GT..9) NGP=10
          IF(FACTOR.GT.1.) NGP=8
          IF(FACTOR.GT.2.) NGP=6
          IF(FACTOR.GT.3.) NGP=5
          IF(FACTOR.GT.4.) NGP=4

*         Selection has been restricted to 12 Gaussian points.
*         If variable Gaussian points are needed , suitable modifications
*         are needed.
.
          NGP = 12
          GFP(1)=0.1252334085       ;WT(1)=0.2491470458
          GFP(2)=-GFP(1)            ;WT(2)=WT(1)
          GFP(3)=0.3678314989982    ;WT(3)=0.23349253654
          GFP(4)=-GFP(3)            ;WT(4)=WT(3)
          GFP(5)=0.5873179543       ;WT(5)=0.203167426723
          GFP(6)=-GFP(5)            ;WT(6)=WT(5)
          GFP(7)=0.7699026742       ;WT(7)=0.160078328543
          GFP(8)=-GFP(7)            ;WT(8)=WT(7)
          GFP(9)=0.9041172563 7     ;WT(9)=0.106939325995
          GFP(20)=-GFP(9)           ;WT(20)=WT(9)
          GFP(11)=0.9815606342467   ;WT(11)=0.0471753363865
          GFP(12)=-GFP(11)                 ;WT(12)=WT(11)
*         Following NAG routine gives the Gaussian points & the weights.
.
*         CALL D01BBF(D01BAZ,-1.,1.,1,NGP,WT,GFP,0)
.
          RETURN;END
C*****************************************************************
*         Formula given by Smith & Liu are used for calculating
*         stresses at the internal points.
.
          SUBROUTINE EXACT(CONLEN,PEAK,XI,YI,NI)
*         IMPLICIT REAL*8(A-H,O-Z)
.         DIMENSION XI(20),YI(20),GFP(32),WT(32)
          COMMON/A/GFP,WT,PI

          WRITE(24,10)
10        FORMAT(10X,'EXACT SOLN.')

          DO 20 I=1,NI
*                   YI(I)=YI(I)-0.036295664
.         AK1=(CONLEN+XI(I))**2+YI(I)*YI(I)
          AK2=(CONLEN-XI(I))**2+YI(I)*YI(I)
          FAC=SQRT(AK2/AK1)
          DEN=FAC*SQRT(2.*FAC+(AK2+AK1-4.*CONLEN**2)/AK1)
          PH1=PI*(1.-FAC)/(AK1*DEN)
          PH2=PI*(1.+FAC)/(AK1*DEN)

          STR11=(-PEAK*YI(I)/PI)*((CONLEN**2+2.*XI(I)*XI(I)+
     1           2.*YI(I)*YI(I))*PH2/CONLEN
     2           -2.*PI/CONLEN-3.*XI(I)*PH1)
          STR22=(-PEAK*YI(I)/PI)*(CONLEN*PH2-XI(I)*PH1)
```

```
        STR22=-PEAK*YI(I)*XI(I)*PHI/PI

        WRITE(24,15)I,STR11,STR12,STR22
15      FORMAT(5X,I4,3E18.8)
20      CONTINUE
        RETURN;END
***********************************************************************
*                                                                     *
*  All the routines that follow are for constant elements.            *
*                                                                     *
***********************************************************************

*       INTRNL calculates stresses at the internal points.

        SUBROUTINE INTRNL(NI,XI,YI,XE,YE,TRACT,DISPL,NC,M,NE)
*       IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION XI(20),YI(20),XE(97),YE(97),TRACT(193),DISPL(193),
     1            NC(2),GFP(32),WT(32)
        COMMON/B/GE,POI
        COMMON/A/GFP,WT,PI

        WRITE(5,790)
790     FORMAT(10X,'PROGRAM IN INTRNL SUBROUTINE')

        IF(NI.EQ.0)GO TO 105

        DO 105 I=1,NI
        ZMA11=0.;ZMA12=0.;ZMA22=0.
**?|    IF NUMBER OF SURFACES CHANGE FOLLOWING STATEMENT
**?|    SHOULD BE CHANGED ACCORDINGLY
        DO 100 J=1,NC(1)
C       WRITE(5,*)I,J
        IF (M-1) 301,301,302
302     IF(J-NC(1))303,304,303
304     TEMP1=XE(J+1);TEMP2=YE(J+1)
        XE(J+1)=XE(1);YE(J+1)=YE(1)
        GO TO 305
303     DO 306 K=2,M
        IF(J-NC(K))306,307,306
307     TEMP1=XE(NC(K)+1);TEMP2=YE(NC(K)+1)
        XE(J+1)=XE(NC(K-1)+1)
        YE(J+1)=YE(NC(K-1)+1)
        GO TO 305
306     CONTINUE
        GO TO 305
301     XE(NE+1)=XE(1);YE(NE+1)=YE(1)
305     CALL STRESS(XI(I),YI(I),XE(J),YE(J),XE(J+1),YE(J+1),
        1 D111,D211,D112,D212,D122,D222,S111,S211,S112,S212,S122,S222,
        2  POI,GE)
        DO 308 KK=1,M
        IF(J.EQ.NC(KK))GO TO 309
        GO TO 308
309     XE(J+1)=TEMP1;YE(J+1)=TEMP2
308     CONTINUE
        ZMA11=ZMA11+D111*TRACT(2*J-1)+D211*
        1 TRACT(2*J)-S111*DISPL(2*J-1)-S211*DISPL(2*J)
        ZMA12=ZMA12+D112*TRACT(2*J-1)+D212*TRACT(2*J)
        1 -S112*DISPL(2*J-1)-S212*DISPL(2*J)
        ZMA22=ZMA22+D122*TRACT(2*J-1)+D222*TRACT(2*J)
        1 -S122*DISPL(2*J-1)-S222*DISPL(2*J)
100     CONTINUE

        WRITE(24,400)I,ZMA11,ZMA12,ZMA22
```

```
400       FORMAT(10X,I4,5X,3E17.8)

105       CONTINUE

*         CAUTION : If stresses at the boundary are needed suitable
*                   modifications are needed. As it is BNDRY
*                   can be used only for linear elements.
.
          DO 10 I=1,NE
             ND=1
          CALL BNDRY(XE,YE,TRACT,DISPL,NC)
          WRITE(24,400) ND,BOUN11,BOUN12,BOUN22
10        CONTINUE

          RETURN;END
C*        *******************************************************************

*         STRESS gives the elemenetal contribution for stresses at
*         internal points.
.
          SUBROUTINE STRESS(XN,YN,XE1,YE1,XE2,YE2,
       1  D111,D211,D112,D212,D122,D222,S111,S211,S112,S212,S122,S222,
       2  POI,GE)
*         IMPLICIT REAL*8(A-H,O-Z)
.         DIMENSION GFP(32),WT(32),XW(32),YW(32)
       1  ,D(2,2),ANORM(2)
          COMMON/A/GFP,WT,PI


C         WRITE(20,50) (XE1,YE1,XE2,YE2,XN,YN)
50        FORMAT(6F10.3)
          ELENGT=SQRT((XE2-XE1)**2+(YE2-YE1)**2)
          ANORM(1)=(YE2-YE1)/ELENGT
          ANORM(2)=(XE1-XE2)/ELENGT
          D111=0.;D211=0.;D112=0.;D212=0.;D122=0.;D222=0.
          S111=0.;S211=0.;S112=0.;S212=0.;S122=0.;S222=0.

                    CALL GAUSS(XN,YN,XE1,YE1,XE2,YE2,NGP,GFP,WT)

          DO 20 K=1,NGP
          XW(K)=(XE2+XE1)/2.+GFP(K)*(XE2-XE1)/2.0
          YW(K)=(YE2+YE1)/2.+GFP(K)*(YE2-YE1)/2.0
          RDIS=SQRT((XN-XW(K))**2+(YN-YW(K))**2)
          R1=(XW(K)-XN)/RDIS
          R2=(YW(K)-YN)/RDIS
C         PAUSE
          DRDN=R1*ANORM(1)+R2*ANORM(2)
          F=2.*GE/(RDIS**2*4.*PI*(1.-POI))
          C=1./(4.*PI*(1.-POI)*RDIS)
          E=1.-2.*POI
          D111=D111+C*(E*R1+2.*R1**3)*WT(K)*ELENGT/2.
          D211=D211+C*((-E)*R2+2.*R1**2*R2)*WT(K)*ELENGT/2.
          D112=D112+C*(E*R2+2.*R1**2*R2)*WT(K)*ELENGT/2.
          D212=D212+C*(E*R1+2.*R2**2*R1)*WT(K)*ELENGT/2.
          D122=D122+C*((-E)*R1+2.*R2**2*R1)*WT(K)*ELENGT/2.
          D222=D222+C*(E*R2+2.*R2**3)*WT(K)*ELENGT/2.
          S111=S111+F*(2.*DRDN*(E*R1+POI*2.*R1-4.*R1**3)+2.*POI*(2.*
       1  ANORM(1)*R1**2)+E*(2.*ANORM(1)*R1**2+2.*ANORM(1))-
       2  (1.-4.*POI)*ANORM(1))*WT(K)*ELENGT/2.
          S211=S211+F*(2.*DRDN*(E*R2-4.*R1**2*R2)+4.*POI*ANORM(1)*R1*R2
       1  +2.*E*ANORM(2)*R1**2-(1.-4.*POI)*ANORM(2))*WT(K)*ELENGT/2.
          S112=S112+F*(2.*DRDN*(POI*R2-4.*R1**2*R2)+2.*POI*(ANORM(1)*
       1  R1*R2+ANORM(2)*R1**2)+E*(2.*ANORM(1)*R1*R2+ANORM(2)))*
       2  WT(K)*ELENGT/2.
```

```
      S212=S212+F*(2.*DRDN*(POI*R1-4.*R1*R2**2)+2.*PUI*(ANORM(1)*
     1 R2**2+ANORM(2)*R1*R2)+E*(2.*ANORM(2)*R1*R2+ANORM(1)))*
     2 WT(K)*ELENGT/2.
      S122=S122+F*(2.*DRDN*(E*P1-4.*R1*R2**2)+4.*POI*ANORM(2)*R1*R2+
     1 2.*E*ANORM(1)*R2**2-(1.-4.*POI)*ANORM(1))*WT(K)*ELENGT/2.
      S222=S222+F*(2.*DRDN*(E*R2+POI*2.*R2-4.*R2**3)+2.*POI*(2.*
     1 ANORM(2)*R2**2)+E*(2.*ANORM(2)*R2**2+2.*ANORM(2))-
     2 (1.-4.*POI)*ANORM(2))*WT(K)*ELENGT/2.
C     WRITE(20,*)(D111,D211,D112,D212,D122,D222)
C     WRITE(20,*)(S111,S211,S112,S212,S122,S222)
20    CONTINUE
      RETURN;END
*     ****************************************************************
*     Internal displacements are calculated in INTU

      SUBROUTINE INTU(NI,NE,NC,XE,YE,XI,YI,POI,GE,TRACT,DISPL,M)

*     IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION XE(97),YE(97),XI(20),YI(20),TRACT(193),DISPL(193),
     1           NC(2),DISINT(20),GEP(32),WT(32)


      COMMON/Z/GEP,WT,PI


      C1=1.-2.*POI
      C2=4.*PI*(1.-POI)
      C3=3.-4.*POI
      C4=8.*PI*GE*(1.-POI)

      DO 80 I=1,NI
      DISINT(2*I-1)=0.
      DISINT(2*I)=0.
      DO 80 J=1,NC(1)
      IF (M-1) 107,107,109
109   IF(J-NC(1))110,111,110
111   TEMP1=XE(J+1);TEMP2=YE(J+1)
      XE(J+1)=XE(1);YE(J+1)=YE(1)
      GO TO 113
110   DO 115 K=2,M
      IF(J-NC(K))115,114,115
114   TEMP1=XE(NC(K)+1);TEMP2=YE(NC(K)+1)
      XE(J+1)=XE(NC(K-1)+1)
      YE(J+1)=YE(NC(K-1)+1)
      GO TO 113
115   CONTINUE
      GO TO 113
107   XE(NE+1)=XE(1);YE(NE+1)=YE(1)
113   CALL GHVALU(XI(I),YI(I),XE(J),YE(J),XE(J+1),YE(J+1),POI,GE,
      1 H11,H12,H21,H22,G11,G12,G21,G22,C1,C2,C3,C4)
      DO 116 KK=1,M
      IF(J.EQ.NC(KK))GO TO 117
      GO TO 116
117   XE(J+1)=TEMP1;YE(J+1)=TEMP2
116   CONTINUE
      DISINT(2*I-1)=DISINT(2*I-1)+TRACT(2*J-1)*G11+TRACT(2*J)*G12
      1                     -DISPL(2*J-1)*H11-DISPL(2*J)*H12
      DISINT(2*I)=DISINT(2*I)+TRACT(2*J-1)*G12+TRACT(2*J)*G22
      1                     -DISPL(2*J-1)*H21-DISPL(2*J)*H22
80    CONTINUE
      WRITE(20,82)(DISINT(2*I-1),DISINT(2*I),I=1,NI)
82    FORMAT(10X,E13.5,5X,E13.5)
```

```
        RETURN;END
*************************************************************************
*
*       GHVALU supplies the elemental contributions for internal
*       displacements. This routine is same as the one used for
*       finding  out [H] & [G] matrices.(in constant elements)
*

        SUBROUTINE GHVALU(XN,YN,XE,YE,XE2,YE2,
     1  PDI,GE,H11,H12,H21,H22,G11,G12,G21,G22,C1,C2,C3,C4)
*       IMPLICIT REAL*8(A-H,O-Z)

        DIMENSION GFP(32),WT(32),XW(32),YW(32)

        COMMON/A/GFP,WT,PI


        XEADD=XE2+XE;  YEADD=YE2+YE
        XEDIF=XE2-XE;  YEDIF=YE2-YE

            ELENGT=SQRT((XEDIF)**2+(YEDIF)**2)

                PRDIS=ABS((YEDIF)*XN-(XEDIF)*YN-
     1          (YE2*XE-YE*XE2))/ELENGT
                SIG=(XE-XN)*(YE2-YN)-(XE2-XN)*(YE-YN)
           IF(SIG.LT.0.)PRDIS=-PRDIS

        H11=0.;H12=0.;H21=0.;H22=0.
        G11=0.;G12=0.;G21=0.;G22=0.

                CALL GAUSS(XN,YN,XE,YE,XE2,YE2,NGP,GFP,WT)

C       TYPE*,NGP
        DO 20 K=1,NGP
        XW(K)=(XEADD)/2.+GFP(K)*(XEDIF)/2.0
        YW(K)=(YEADD)/2.+GFP(K)*(YEDIF)/2.0
        RDIS=SQRT((XN-XW(K))**2+(YN-YW(K))**2)
        R1=(XW(K)-XN)/RDIS
        R2=(YW(K)-YN)/RDIS
        XNORM=(YEDIF)/ELENGT
        YNORM=-XEDIF/ELENGT

        WTH=WT(K)*ELENGT/(2.*C2*RDIS)
        RATIO=PRDIS/RDIS
        WTG=WT(K)*ELENGT/(2.*C4)
        H11=H11-RATIO*(C1+2.*R1**2)*WTH
        H12=H12-(RATIO*(2.*R1*R2)-(C1*
     1  (R1*YNORM-R2*XNORM)))*WTH
        H21=H21-(RATIO*(2.*R1*R2)-C1*
     1  (R2*XNORM-R1*YNORM))*WTH
        H22=H22-RATIO*(C1+2.*R2**2)*WTH
        G11=G11+(C3*ALOG(1./RDIS)+R1*R1)*WTG
        G12=G12+R1*R2*WTG
        G21=G12
        G22=G22+(C3*ALOG(1./RDIS)+R2*R2)*WTG
20      CONTINUE
        RETURN;END
*************************************************************************
```

Details about input data.

Datas are for [illegible] cylinder over a steel foundation.

All the values are given in free format.

Coordinates are for radius = 50.

Foundation dimensions are selected in such a manner that it most behaves like infinite foundation. When radius of the wheel anges ,foundation dimensions are also changed by the same factor.

In the contact region elemental lengts are very small when mpared with the normal nodes. This to define the contact region hich itself is small) better.

```
 0.0000000D+00      2
95      10    0.1692600D+04    0.7656700D+05
 0.2562560D+00    0.2562560D+00    0.3000000D+01    15    1
 0.4000000D+00
 6
 1     2
 3     4
 9    10
11    12
49    50
51    52
56
95
-0.3600000D+01    0.5144231D+01
-0.3200000D+01    0.1012821D+02
-0.2800000D+01    0.1511218D+02
-0.2400000D+01    0.2009615D+02
-0.2000000D+01    0.2508013D+02
-0.1600000D+01    0.3006410D+02
-0.1200000D+01    0.3504808D+02
-0.8000000D+00    0.4003225D+02
-0.4000000D+00    0.4501603D+02
 0.0000000D+00    0.5000000D+02
 0.4357787D+01    0.9980974D+02
 0.4357787D+01    0.9980974D+02
-0.4357787D+01    0.9980974D+02
-0.4357787D+01    0.9980974D+02
-0.1657791D+02    0.9717174D+02
-0.3128035D+02    0.8900692D+02
-0.4244404D+02    0.7642922D+02
-0.4880601D+02    0.6086158D+02
-0.4980973D+02    0.5435779D+02
-0.4980973D+02    0.5435779D+02
-0.4980973D+02    0.4564221D+02
-0.4980973D+02    0.4564221D+02
-0.4487051D+02    0.2794015D+02
-0.3501828D+02    0.1431078D+02
-0.2120441D+02    0.4718954D+01
-0.1710101D+02    0.3015369D+01
-0.1294095D+02    0.1703709D+01
-0.8682409D+01    0.7596123D+00
-0.4357787D+01    0.1902655D+00
-0.3487824D+01    0.1217984D+00
-0.1999467D+01    0.3999509D-01
-0.1799611D+01    0.3239699D-01
-0.1599727D+01    0.2559833D-01
-0.1399817D+01    0.1959875D-01
-0.1199885D+01    0.1440011D-01
-0.9999332D+00    0.9999797D-02
-0.7999655D+00    0.6400049D-02
-0.5999855D+00    0.3600493D-02
-0.3999957D+00    0.1600012D-02
-0.1999994D+00    0.4000962D-03
 0.1999994D+00    0.4000962D-03
 0.3999957D+00    0.1600012D-02
 0.5999855D+00    0.3600493D-02
 0.7999655D+00    0.6400049D-02
 0.9999332D+00    0.9999797D-02
 0.1199885D+01    0.1440011D-01
 0.1399817D+01    0.1959875D-01
 0.1599727D+01    0.2559833D-01
 0.1799611D+01    0.3239699D-01
 0.1999467D+01    0.3999509D-01
 0.3487824D+01    0.1217984D+00
```

0.4357787D+01	0.1902655D+00
0.8682409D+01	0.7596123D+00
0.1294095D+02	0.1703709D+01
0.1710101D+02	0.3015369D+01
0.2120141D+02	0.4719954D+01
0.3501428D+02	0.1431078D+02
0.4487051D+02	0.2794015D+02
0.4980973D+02	0.4564221D+02
0.4980973D+02	0.4564221D+02
0.4980973D+02	0.5435779D+02
0.4980973D+02	0.5435779D+02
0.4886601D+02	0.6086158D+02
0.4244404D+02	0.7642922D+02
0.3128035D+02	0.8900692D+02
0.1657791D+02	0.9717174D+02
0.1000000D+03	0.0000000D+00
0.5000000D+02	0.0000000D+00
0.1745329D+02	0.0000000D+00
0.1308997D+02	0.0000000D+00
0.8726646D+01	0.0000000D+00
0.4563324D+01	0.0000000D+00
0.3690659D+01	0.0000000D+00
0.2000000D+01	0.0000000D+00
0.1800000D+01	0.0000000D+00
0.1600000D+01	0.0000000D+00
0.1400000D+01	0.0000000D+00
0.1200000D+01	0.0000000D+00
0.1000000D+01	0.0000000D+00
0.8000000D+00	0.0000000D+00
0.6000000D+00	0.0000000D+00
0.4000000D+00	0.0000000D+00
0.2000000D+00	0.0000000D+00
-0.2000000D+00	0.0000000D+00
-0.4000000D+00	0.0000000D+00
-0.6000000D+00	0.0000000D+00
-0.8000000D+00	0.0000000D+00
-0.1000000D+01	0.0000000D+00
-0.1200000D+01	0.0000000D+00
-0.1400000D+01	0.0000000D+00
-0.1600000D+01	0.0000000D+00
-0.1800000D+01	0.0000000D+00
-0.2000000D+01	0.0000000D+00
-0.3690659D+01	0.0000000D+00
-0.4563324D+01	0.0000000D+00
-0.8726646D+01	0.0000000D+00
-0.1308997D+02	0.0000000D+00
-0.1745329D+02	0.0000000D+00
-0.5000000D+02	0.0000000D+00
-0.1000000D+03	0.0000000D+00
-0.1000000D+03	-0.7500000D+02
-0.1000000D+03	-0.1500000D+03
0.0000000D+00	-0.1500000D+03
0.1000000D+03	-0.1500000D+03
0.1000000D+03	-0.7500000D+02

1	0.0000000D+00	1	0.0000000D+00	1
2	0.0000000D+00	1	-0.1200000D+02	1
3	0.0000000D+00	1	-0.1200000D+02	1
4	0.0000000D+00	1	0.0000000D+00	1
5	0.0000000D+00	1	0.0000000D+00	1
6	0.0000000D+00	1	0.0000000D+00	1
7	0.0000000D+00	1	0.0000000D+00	1
8	0.0000000D+00	1	0.0000000D+00	1
9	0.0000000D+00	1	0.0000000D+00	1
10	-0.5000000D-01	1	0.0000000D+00	1

```
          75    0.0000000D+00      1    0.0000000D+00      1
          76    0.0000000D+00      1    0.0000000D+00      1
          77    0.0000000D+00      1    0.0000000D+00      1
          78    0.0000000D+00      1    0.0000000D+00      1
          79    0.0000000D+00      1    0.0000000D+00      1
          80    0.0000000D+00      1    0.0000000D+00      1
          81    0.0000000D+00      1    0.0000000D+00      1
          82    0.0000000D+00      1    0.0000000D+00      1
          83    0.0000000D+00      1    0.0000000D+00      1
          84    0.0000000D+00      1    0.0000000D+00      1
          85    0.0000000D+00      1    0.0000000D+00      1
          86    0.0000000D+00      1    0.0000000D+00      1
          87    0.0000000D+00      1    0.0000000D+00      1
          88    0.0000000D+00      1    0.0000000D+00      1
          89    0.0000000D+00      1    0.0000000D+00      1
          90    0.0000000D+00      1    0.0000000D+00      1
          91    0.0000000D+00      1    0.0000000D+00      1
          92    0.0000000D+00      0    0.0000000D+00      0
          93    0.0000000D+00      0    0.0000000D+00      0
          94    0.0000000D+00      0    0.0000000D+00      0
          95    0.0000000D+00      1    0.0000000D+00      1
74  30  75  29  76  28  77  27  78  26
79  25  80  24  81  23  82  22  83  21
84  20  85  19  86  18  87  17  88  16
73  31  72  32  71  33  70  34  69  35
68  36  67  37  66  38  65  39  64  40
63  41  62  42  61  43  60  44  59  45
   0.2291831D+00    0.4583662D+00    0.6875493D+00    0.9167324D+00
   0.1145915D+01    0.1375099D+01    0.1604282D+01    0.1833465D+01
   0.2062648D+01    0.2291831D+01    0.4000000D+01    0.5000000D+01
   0.1000000D+02    0.1500000D+02    0.2000000D+02
   0.2291831D+00    0.4583662D+00    0.6875493D+00    0.9167324D+00
   0.1145915D+01    0.1375099D+01    0.1604282D+01    0.1833465D+01
   0.2062648D+01    0.2291831D+01    0.4000000D+01    0.5000000D+01
   0.1000000D+02    0.1500000D+02    0.2000000D+02
         6       6       6       6       6       1
```

Results from CONTACT Program (Page 1 to 12)

E NO.	DISPLACEMENT X	DISPLACEMENT Y	TRACTION X	TRACTION Y
1	-0.58299E+00	-0.12291E+00	0.00000E+00	0.00000E+00
2	-0.58299E+00	-0.12291E+00	0.00000E+00	-0.12000E+02
3	-0.57544E+00	-0.17403E+00	0.00000E+00	-0.12000E+02
4	-0.57544E+00	-0.17403E+00	0.00000E+00	0.00000E+00
5	-0.55882E+00	-0.21273E+00	0.00000E+00	0.00000E+00
6	-0.51783E+00	-0.28638E+00	0.00000E+00	0.00000E+00
7	-0.44909E+00	-0.34750E+00	0.00000E+00	0.00000E+00
8	-0.36069E+00	-0.38365E+00	0.00000E+00	0.00000E+00
9	-0.32311E+00	-0.38957E+00	0.00000E+00	0.00000E+00
10	-0.32311E+00	-0.38957E+00	-0.50000E-01	0.00000E+00
11	-0.27199E+00	-0.38970E+00	-0.50000E-01	0.00000E+00
12	-0.27199E+00	-0.38970E+00	0.00000E+00	0.00000E+00
13	-0.16589E+00	-0.36009E+00	0.00000E+00	0.00000E+00
14	-0.81671E-01	-0.29918E+00	0.00000E+00	0.00000E+00
15	-0.19823E-01	-0.20922E+00	0.00000E+00	0.00000E+00
16	-0.79944E-02	-0.18101E+00	0.00000E+00	0.00000E+00
17	0.15083E-02	-0.15094E+00	0.00000E+00	0.00000E+00
18	0.88537E-02	-0.11751E+00	0.00000E+00	0.00000E+00
19	0.13247E-01	-0.77104E-01	0.00000E+00	0.00000E+00
20	0.14055E-01	-0.66935E-01	0.00000E+00	0.00000E+00
21	0.13847E-01	-0.44724E-01	0.00000E+00	0.00000E+00
22	0.13788E-01	-0.40701E-01	0.00000E+00	0.00000E+00
23	0.13520E-01	-0.36064E-01	0.00000E+00	0.00000E+00
24	0.12830E-01	-0.30346E-01	-0.21270E+00	0.53175E+00
25	0.10989E-01	-0.23008E-01	-0.11906E+02	0.29765E+02
26	0.89913E-02	-0.16807E-01	-0.15794E+02	0.39485E+02
27	0.73899E-02	-0.11740E-01	-0.19527E+02	0.48817E+02
28	0.67345E-02	-0.77991E-02	-0.13642E+02	0.55643E+02
29	0.66352E-02	-0.49718E-02	-0.49890E+01	0.58802E+02
30	0.65300E-02	-0.32593E-02	0.60004E+00	0.59940E+02
31	0.63377E-02	-0.31844E-02	0.14075E+02	0.55698E+02
32	0.54625E-02	-0.48204E-02	0.19257E+02	0.48143E+02
33	0.37396E-02	-0.75830E-02	0.15603E+02	0.39008E+02
34	0.16425E-02	-0.11478E-01	0.11713E+02	0.29283E+02
35	-0.28106E-03	-0.16514E-01	-0.34051E+00	-0.85129E+00
36	-0.10197E-02	-0.19800E-01	0.00000E+00	0.00000E+00
37	-0.13936E-02	-0.22036E-01	0.00000E+00	0.00000E+00
38	-0.16369E-02	-0.23702E-01	0.00000E+00	0.00000E+00
39	-0.17963E-02	-0.24985E-01	0.00000E+00	0.00000E+00
40	-0.18670E-02	-0.25971E-01	0.00000E+00	0.00000E+00
41	-0.26797E-02	-0.28520E-01	0.00000E+00	0.00000E+00
42	-0.27181E-02	-0.27936E-01	0.00000E+00	0.00000E+00
43	-0.50124E-02	-0.16584E-01	0.00000E+00	0.00000E+00
44	-0.88181E-02	0.28727E-03	0.00000E+00	0.00000E+00
45	-0.14756E-01	0.19179E-01	0.00000E+00	0.00000E+00
46	-0.22955E-01	0.39201E-01	0.00000E+00	0.00000E+00
47	-0.73727E-01	0.11146E+00	0.00000E+00	0.00000E+00
48	-0.14948E+00	0.16619E+00	0.00000E+00	0.00000E+00
49	-0.25123E+00	0.19452E+00	0.00000E+00	0.00000E+00
50	-0.25123E+00	0.19452E+00	0.00000E+00	0.00000E+00
51	-0.30236E+00	0.19462E+00	0.00000E+00	0.00000E+00
52	-0.30236E+00	0.19462E+00	0.00000E+00	0.00000E+00
53	-0.34098E+00	0.18876E+00	0.00000E+00	0.00000E+00
54	-0.43512E+00	0.15031E+00	0.00000E+00	0.00000E+00
55	-0.51389E+00	0.80499E-01	0.00000E+00	0.00000E+00
56	-0.56867E+00	-0.18292E-01	0.00000E+00	0.00000E+00
57	-0.10412E-03	-0.90102E-04	0.00000E+00	0.00000E+00
58	-0.15735E-03	-0.34838E-03	0.00000E+00	0.00000E+00
59	-0.20480E-03	-0.79869E-03	0.00000E+00	0.00000E+00
60	-0.22345E-03	-0.92692E-03	0.00000E+00	0.00000E+00
61	-0.24041E-03	-0.11113E-02	0.00000E+00	0.00000E+00
62	-0.25405E-03	-0.14040E-02	0.00000E+00	0.00000E+00
3	-0.26484E-03	-0.14992E-02	0.00000E+00	0.00000E+00
	-0.27971E-03	-0.17754E-02	0.00000E+00	0.00000E+00

65	-0.28623E-03	-0.18227E-02	0.00000E+00	0.00000E+00
66	-0.29291E-03	-0.18769E-02	0.00000E+00	0.00000E+00
67	-0.30101E-03	-0.19395E-02	0.00000E+00	0.00000E+00
68	-0.31196E-03	-0.20147E-02	0.00000E+00	0.00000E+00
69	-0.32832E-03	-0.21142E-02	0.34051E+00	0.85129E+00
70	-0.34608E-03	-0.22622E-02	-0.11713E+02	-0.29283E+02
71	-0.33443E-03	-0.23983E-02	-0.15603E+02	-0.39008E+02
72	-0.29490E-03	-0.25163E-02	-0.19257E+02	-0.48143E+02
73	-0.22005E-03	-0.26082E-02	-0.14075E+02	-0.55698E+02
74	-0.27722E-04	-0.26832E-02	-0.60004E+00	-0.59940E+02
75	0.77478E-04	-0.26678E-02	0.49890E+01	-0.58802E+02
76	0.17676E-03	-0.26144E-02	0.43642E+02	-0.55643E+02
77	0.25425E-03	-0.25243E-02	0.49527E+02	-0.48817E+02
78	0.29602E-03	-0.24070E-02	0.15794E+02	-0.39485E+02
79	0.30944E-03	-0.22716E-02	0.11906E+02	-0.29765E+02
80	0.29342E-03	-0.21240E-02	0.21270E+00	-0.53175E+00
81	0.27716E-03	-0.20223E-02	0.00000E+00	0.00000E+00
82	0.26567E-03	-0.19460E-02	0.00000E+00	0.00000E+00
83	0.25583E-03	-0.18837E-02	0.00000E+00	0.00000E+00
84	0.24012E-03	-0.15546E-02	0.00000E+00	0.00000E+00
85	0.22804E-03	-0.14487E-02	0.00000E+00	0.00000E+00
86	0.21653E-03	-0.11356E-02	0.00000E+00	0.00000E+00
87	0.20054E-03	-0.94377E-03	0.00000E+00	0.00000E+00
88	0.18256E-03	-0.81197E-03	0.00000E+00	0.00000E+00
89	0.13860E-03	-0.35601E-03	0.00000E+00	0.00000E+00
90	0.85380E-04	-0.10048E-03	0.00000E+00	0.00000E+00
91	-0.14369E-03	-0.12573E-03	0.00000E+00	0.00000E+00
92	0.00000E+00	0.00000E+00	0.17978E+00	0.23115E+00
93	0.00000E+00	0.00000E+00	0.85725E-03	0.64858E+00
94	0.00000E+00	0.00000E+00	-0.17580E+00	0.21984E+00
95	0.13392E-03	-0.11731E-03	0.00000E+00	0.00000E+00

Results from the second Program (Page 13 to 22)

(Stresses at internal points)

ODE	X-COORD	Y-COORD	SIGMA-X	SIGMA-XY	SIGMA-Y
1	-0.50000E+01	0.20000E+01	-0.65255E+01	0.32421E+01	-0.16065E+01
2	0.50000E+01	0.20900E+01	-0.55523E+01	-0.23836E+01	-0.10020E+01
3	-0.50000E+01	0.40000E+01	-0.53275E+01	0.49685E+01	-0.49310E+01
4	0.50000E+01	0.40000E+01	-0.51853E+01	-0.42778E+01	-0.37295E+01
5	-0.50000E+01	0.60000E+01	-0.32030E+01	0.45483E+01	-0.65388E+01
6	0.50000E+01	0.60000E+01	-0.34189E+01	-0.43065E+01	-0.54614E+01
7	-0.50000E+01	0.80000E+01	-0.17830E+01	0.36509E+01	-0.68418E+01
8	0.50000E+01	0.80000E+01	-0.20408E+01	-0.36575E+01	-0.60522E+01
9	-0.50000E+01	0.10000E+02	-0.94216E+00	0.28484E+01	-0.65848E+01
10	0.50000E+01	0.10000E+02	-0.11579E+01	-0.29555E+01	-0.60333E+01
11	-0.50000E+01	0.15000E+02	-0.25147E-01	0.15858E+01	-0.54476E+01
12	0.50000E+01	0.15000E+02	-0.13366E+00	-0.17208E+01	-0.52146E+01
13	0.00000E+00	0.20000E+01	-0.45225E+01	-0.26615E+01	-0.33441E+02
14	0.00000E+00	0.40000E+01	-0.85174E+00	-0.12670E+01	-0.21090E+02
15	0.00000E+00	0.60000E+01	-0.90565E-01	-0.67285E+00	-0.15037E+02
16	0.00000E+00	0.80000E+01	0.17908E+00	-0.40857E+00	-0.11629E+02
17	0.00000E+00	0.10000E+02	0.31102E+00	-0.27272E+00	-0.94755E+01
18	0.00000E+00	0.15000E+02	0.45636E+00	-0.12782E+00	-0.64943E+01
19	0.00000E+00	0.20000E+02	0.51336E+00	-0.73942E-01	-0.49675E+01
20	0.00000E+00	0.25000E+02	0.54148E+00	-0.48353E-01	-0.40505E+01